Chapter 4
Monetary Policy (I)

This chapter provides a first half of our discussions on monetary policy. First, we extend our analysis on fiscal policy to introduce seigniorage and see how seigniorage may cause inflation. Second, we provide a theoretical model on how money growth affects the output of an economy.

4.1 Seigniorage and Inflation

Question: *Why May Issuing Debts and Printing Money to Finance Government Deficits Bring Some Unusual Episodes of Hyperinflation? That is, Why Are Balanced Budgets Important? (Reference: Chapter 18 of DLS.)*

The previous two chapters discussed two ways to finance government spending (G_t): (i) collecting taxes (T_t) and (ii) issuing government debts (B_t, which denotes a one-period debt). At time t, the government constraint used to be

$$\underbrace{G_t + (1+r)B_{t-1}}_{\text{to be financed}} = T_t + B_t, \tag{4.1}$$

for all $t = 0, 1, 2, ...$, where $B_{-1} = 0$ is given (i.e., there is no government debt to be paid at period 0). The goal of this section is to introduce a third way: (iii) printing money.

Since "money" is now introduced for the first time in this course, we have to clearly distinguish *real* terms from *nominal* terms. Assume there is only one type of consumption goods (or "apples"). In (4.1), all variables, G_t, B_t and T_t, are in units of consumption goods, i.e., in **real** terms. Now, let M_t denote the amount of money existing in this economy, in dollars, i.e., in **nominal** terms, at the end of period t. This means that the central bank prints $M_t - M_{t-1}$ dollars during period t. If this amount of dollars are provided to the government, the government can additionally buy

$$\frac{M_t - M_{t-1}}{P_t} \tag{4.2}$$

units of consumption goods, where P_t is the nominal price of one unit of consumption good at period t. The amount (4.2), which is the units of government spending financed by printing money, is called **seigniorage**. With seigniorage, the government constraint (4.1) is replaced by a new equation:

$$\underbrace{G_t + (1+r)B_{t-1}}_{\text{to be financed}} = \underbrace{T_t}_{\text{by taxation}} + \underbrace{B_t}_{\text{by borrowing}} + \underbrace{\frac{M_t - M_{t-1}}{P_t}}_{\text{by printing money}}, \tag{4.3}$$

for all $t = 0, 1, 2, ...$, where $B_{-1} = 0$ is given. Notice that the government spending financed by seigniorage ultimately comes from consumers. Consumers eventually finance the seigniorage.

Printing money causes **inflation** (an increase in the overall price level), which sometimes develops to **hyperinflation**. For example, in Germany of the early 1920s, the price level doubled every 49 hours (i.e., the inflation rate was 3 million % per month). In Hungary after World War II, it doubled every 15 hours (4 quadrillion (10^{16}) % per month). Hyperinflation is destructive: Under hyperinflation, holding money is very costly because the purchasing power decreases, and as a result, everyone is willing to find ways *not* to hold money – for example, hoarding of real assets or extreme consumption.

But isn't money supply controllable by the central bank? Why do the central bank want to print such an incredible amount of money only to bring hyperinflation? The following model, often called "Some Unpleasant Monetarist Arithmetic" (based on an article by Thomas Sargent and Neil Wallace), will help us to understand it.

4.1.1 Model Description

We assume the following:

(1) Population grows at a constant rate, n. That is,

$$\frac{N_t}{N_{t-1}} = 1 + n \tag{4.4}$$

for all $t = 1, 2, ...$, where N_t denotes the population size at period t.

(2) Output per capita (in real terms, i.e., in units of consumption goods), y_t, is constant and equal to 1. That is,

$$y_t = 1.$$

So total output of the economy in real terms, or real GDP, becomes

$$Y_t = N_t y_t = N_t, \tag{4.5}$$

which grows at n. Remember: N_t is both population size and real GDP.

(3) *Government debt cannot exceed some given level.* In particular, the debt-to-GDP level cannot be higher than $\bar{b}$, i.e.,

$$\frac{B_t}{N_t} \equiv b_t \leq \bar{b}, \tag{4.6}$$

for all $t = 0, 1, ...$

(4) There are positive deficits at all periods (i.e., $G_t - T_t > 0$). In particular, for simplicity, assume

$$\frac{G_t - T_t}{N_t} = d > 0, \tag{4.7}$$

for all $t = 0, 1, ...$ This means that the decifit-to-GDP level is constant over time and positive. (4.7) makes the model interesting: Spendings are not fully financed by taxation, so the government should determine how much to borrow and how much to print.

(5) The **quantity theory of money**,

$$M_t V_t = P_t Y_t, \tag{4.8}$$

holds with a constant velocity, $V_t = 1$. This theory means the following. Y_t is the total output in real terms. So Y_t units of final goods are produced and *purchased* by somebody in the market. Then $P_t Y_t$ dollars should be transacted, which is called money demand. If money can be used only once, this money demand should be equal to M_t, which is money supply. But money is used more than once during a given period. If money turns over twice, we should have $P_t Y_t / 2 = M_t$. Similarly, denoting by V_t the **velocity of money** (average number of times a piece of money turns over in a period), we shoould have $P_t Y_t / V_t = M_t$, or equivalently, (4.8).

In this model, we further assume $V_t = 1$, so

$$P_t Y_t = M_t. \tag{4.9}$$

Then (4.5) and (4.9) imply that

$$P_t = \frac{M_t}{N_t}. \tag{4.10}$$

This equation determines the price level (P_t), given money supply (M_t) and real GDP (or population) (N_t).

(6) Monetary policy of the central bank is to determine the growth rate of money supply. In particular, assume the central bank wants to keep this (net) growth rate constant at θ. Then, the monetary policy is to pick θ in which

$$\frac{M_t}{M_{t-1}} = 1 + \theta, \tag{4.11}$$

whenever money growth is controllable. Notice that

$$\begin{aligned} \frac{M_t - M_{t-1}}{P_t} &= \frac{M_t}{P_t} - \frac{M_{t-1}}{M_t}\frac{M_t}{P_t} \\ &= \frac{M_t}{P_t}\left(1 - \frac{M_{t-1}}{M_t}\right) \\ &= N_t\left(1 - \frac{M_{t-1}}{M_t}\right) \quad \text{from (4.10)} \end{aligned}$$

So (4.3) can be written as

$$G_t + (1+r)B_{t-1} = T_t + B_t + N_t\left(1 - \frac{M_{t-1}}{M_t}\right). \tag{4.12}$$

In particular, under the monetary policy (4.11), this becomes

$$\underbrace{G_t + (1+r)B_{t-1}}_{\text{to be financed}} = \underbrace{T_t}_{\text{by taxation}} + \underbrace{B_t}_{\text{by borrowing}} + \underbrace{N_t\left(\frac{\theta}{1+\theta}\right)}_{\text{by printing money}},$$

that is, *the seigniorage becomes* $N_t\left(\frac{\theta}{1+\theta}\right)$*, or equivalently, a fraction* $\frac{\theta}{1+\theta}$ *of the GDP.*

4.1.2 Solution and Discussion

Now we are ready to see how government debt (B_t), money supply (M_t), and price level (P_t) evolve over periods. First, divide both sides of (4.12) by N_t. That is,

$$\frac{G_t}{N_t} + (1+r)\frac{B_{t-1}}{N_t} = \frac{T_t}{N_t} + \frac{B_t}{N_t} + \left(1 - \frac{M_{t-1}}{M_t}\right).$$

Rearranging,

$$\begin{aligned}\frac{B_t}{N_t} &= (1+r)\frac{B_{t-1}}{N_t} + \frac{G_t - T_t}{N_t} - \left(1 - \frac{M_{t-1}}{M_t}\right) \\ &= (1+r)\frac{N_{t-1}}{N_t}\frac{B_{t-1}}{N_{t-1}} + d - \left(1 - \frac{M_{t-1}}{M_t}\right) \quad \text{by (4.7)} \\ &= \frac{1+r}{1+n}\frac{B_{t-1}}{N_{t-1}} + d - \left(1 - \frac{M_{t-1}}{M_t}\right) \quad \text{by (4.4)}\end{aligned}$$

Hence, the evolution of debt-to-GDP ratio, $b_t \equiv B_t/N_t$, is given by

$$b_t = \frac{1+r}{1+n}b_{t-1} + d - \left(1 - \frac{M_{t-1}}{M_t}\right). \tag{4.13}$$

We see what will happen in this fictional economy.

1. Evolution of Debt/GDP (b_t): Under the monetary policy (4.11), (4.13) becomes

$$b_t = \frac{1+r}{1+n}b_{t-1} + d - \frac{\theta}{1+\theta}.$$

Therefore, we have

$$\begin{aligned}b_0 &= d - \frac{\theta}{1+\theta}, \\ b_1 &= \frac{1+r}{1+n}\left(d - \frac{\theta}{1+\theta}\right) + d - \frac{\theta}{1+\theta} \\ &= \left(1 + \frac{1+r}{1+n}\right)\left(d - \frac{\theta}{1+\theta}\right), \\ b_2 &= \left[1 + \frac{1+r}{1+n} + \left(\frac{1+r}{1+n}\right)^2\right]\left(d - \frac{\theta}{1+\theta}\right),\end{aligned}$$

so

$$\begin{aligned}b_t &= \left(d - \frac{\theta}{1+\theta}\right)\sum_{i=0}^{t}\left(\frac{1+r}{1+n}\right)^i \qquad (4.14) \\ &= \left(d - \frac{1}{1/\theta+1}\right)\sum_{i=0}^{t}\left(\frac{1+r}{1+n}\right)^i \quad \text{if } \theta \neq 0 \qquad (4.15)\end{aligned}$$

for all $t = 0, 1, ...$ in which (4.11) is true. This is an explicit solution for debt-to-

GDP ratio, b_t.

2. Choice of Monetary Policy (θ): Assume $r > n$. (We believe this is true in reality.) The term $\sum_{i=0}^{t}\left(\frac{1+r}{1+n}\right)^i$ in (4.14) or (4.15) is explosive. [Draw a line for θ.] Suppose θ is zero (i.e., the bank does not print any money so there is no money growth). Then, the term $\left(d - \frac{\theta}{1+\theta}\right)$ in (4.14) becomes d. As θ increases from 0 (i.e., as the bank increases money growth), as is clear from (4.15), this term $\left(d - \frac{\theta}{1+\theta}\right)$ decreases, so b_t decreases, which means the government borrows *less* in the financial market. The term $\left(d - \frac{\theta}{1+\theta}\right)$ will reach 0 if θ increases to θ^* in which

$$d - \frac{1}{1/\theta^* + 1} = 0,$$

or equivalently,

$$\theta^* = \frac{1}{1/d - 1}. \tag{4.16}$$

If the central bank increases θ even more from this level, then the term $\left(d - \frac{\theta}{1+\theta}\right)$ becomes negative. But this means that b_t is also negative, so the government is *lending* resources. Rule out this case: Facing the fiscal deficit, the government tries to print money and to *borrow* money (instead of lending), so assume that θ is not higher than θ^* in (4.16).

3. Catastrophe Date (T): Let's go back to (4.14). If $r > n$ and if $\theta \leq \theta^*$, the debt-to-GDP ratio, b_t, grows over periods (except that it stays at 0 when $\theta = \theta^*$) according to (4.14). But wait. *This ratio cannot grow forever* because of (4.6). In other words, when $\theta < \theta^*$ is given, b_t increases until some period T in which b_t reaches $\bar{b}$. After that, b_t *should stay at the same level forever*. This period T is called the **catastrophe period**. [Draw a time path for b_t.] Mathematically, T is roughly obtained by solving:

$$\left(d - \frac{\theta}{1+\theta}\right)\sum_{i=0}^{T}\left(\frac{1+r}{1+n}\right)^i = \bar{b}.$$

4. Money Supply (M_t) and Price Level (P_t): Now consider the money growth, M_t/M_{t-1}. Before T, from (4.11), gross money growth is $1+\theta$. After T, issuing debts is restricted, so the central bank is forced to print a lot of money to generate seigniorage. Formally, since $b_t = \bar{b}$, (4.13) implies that

$$\bar{b} = \frac{1+r}{1+n}\bar{b} + d - \left(1 - \frac{M_{t-1}}{M_t}\right),$$

for $t = T+1, T+2, ...$, or equivalently,

$$\frac{M_t}{M_{t-1}} = \frac{1}{1 - d + \left(1 - \frac{1+r}{1+n}\right) \times \bar{b}},$$

for $t = T+1, T+2, ...$ But

$$\begin{aligned} 1+\theta &\leq 1 + \frac{1}{1/d - 1} \quad \text{from (4.16)} \\ &= \frac{1}{1-d} \\ &< \frac{1}{1 - d + \left(1 - \frac{1+r}{1+n}\right) \times \bar{b}} \quad \text{since } r > n \end{aligned}$$

which means

$$\underbrace{1+\theta}_{\text{money growth before } T} < \underbrace{\frac{1}{1 - d + \left(1 - \frac{1+r}{1+n}\right) \times \bar{b}}}_{\text{money growth after } T}.$$

So the money growth should definitely increase after T. [Draw a time path for M_t/M_{t-1}. Also for M_t. This should be amazing.] Of course, from (4.10), higher money growth implies higher inflation other things being equal. [Draw a time path for P_t.]

Result 1: *If there are fiscal deficits at all periods, they should be financed by accumulating issued debts and/or by printing more money. However, after the debt level reaches the limit, the central bank is forced to increase the money growth because issuing debts becomes difficult. This bring higher, uncontrollable inflation.*

5. Dilemma of the Central Bank: The central bank can choose between lower and higher values of θ (money growth). The second result of this model comes from considering different values on θ:

(1) Lower money growth (lower value of θ) now → Lower inflation before T (which is *good* now) → A larger part of deficit to be financed by borrowing

$\rightarrow$ Debt-to-GDP ratio, b_t, reaching its limit $\bar{b}$ sooner $\rightarrow$ Catastrophe period (T) arriving early (which is *bad* later)

(2) Higher money growth (higher value of θ) now $\rightarrow$ Higher inflation before T (which is *bad* now)$\rightarrow$ A smaller part of deficit to be financed by borrowing $\rightarrow$ Debt-to-GDP ratio, b_t, reaching its limit $\bar{b}$ later $\rightarrow$ Catastrophe period (T) arriving late (which is *good* later)

And, going extreme, we can make the inflation rate constant forever:

(2') Very high money growth ($\theta = \theta^*$) now $\rightarrow$ Much higher inflation $\rightarrow$ No part of deficit to be financed by borrowing $\rightarrow$ Debt-to-GDP ratio, b_t, staying at zero $\rightarrow$ Catastrophe period (T) never arriving $\rightarrow$ Inflation rate staying

Now the conclusion:

Result 2: *If there are fiscal deficits at all periods, the central bank should choose between low inflation today and later onset of hyperinflationary catastrophe.*

6. Balanced Budget: To avoid this "unpleasant" trade-off, one of the assumptions needs to be broken. The most important assumption that brings this dilemma is (4.7). That is, if budgets are balanced, these problems will not arise.

Result 3: *Balanced budgets are important!*

Thomas Sargent, "The Ends of Four Big Inflations," *Inflation: Causes and Effects* (1983), considers four episodes of hyperinflation in Poland, Hungary, Austria and Germany in 1919-1924. He argues that (1) governments were running huge deficits after the war, which were financed by borrowing and printing money, and that (2) hyperinflation ended after regime changes: "Governments abandoned fiscal deficits and seigniorage financing in favor of balanced budgets and independent central banks." (DLS, p. 222) These observations are consistent with our results.

True or False? —

1. Assume that the real per-capita output is constant at one unit of consumption good and that the population grows at n (e.g., 1%) where $n > 0$. Also, suppose that the quantity theory of money holds with a constant velocity. Then, even though the money supply increases, i.e., $M_t/M_{t-1} > 1$, it may happen that the price level decreases, i.e., $P_t/P_{t-1} < 1$.

2. Consider an economy in which the per-capita real output increases by 2% per year, population grows at 3% per year, and money stock (M_t) grows at 6%

per year. If the quantity theory of money holds with a constant velocity of money (V), then the price level of this economy will stay at the same level over time.

3. Consider the environment in "Some Unpleasant Monetarist Arithmetic." Suppose the government spending exceeds the government tax revenue in an economy in which the government is not legally allowed to issue any debts. Then, the central bank in this country is not able to control the inflation.

4.2 Price Level, Output and (Un)Employment ("Monetary Neutrality")

Question: *How Does Money Growth (or Inflation) Affect the Real Output and (Un)Employment Rate? (References: Chapter 19 of DLS and various papers including Lucas's (1995) Nobel lecture, "Monetary Neutrality," Economic Sciences.)*

If the central bank prints more money (and inflation arises), will it increase or decrease the *real* output? To answer this question, we will introduce two theoretical models. In the first, the overall price level is perfectly predictable. In the second, the information on the overall price level is *imperfect*, so consumers are not sure about tomorrow's price level. The conclusion we will derive is that the real output may be affected by "unanticipated" changes in money supply (or in price level).

4.2.1 Model 1: Perfect Information on Price Level

There are N industries, and each industry has one representative consumer. Industry i produces good i with the following (symmetric) production function:

$$Y_i = L_i, \quad \text{for } i = 1, ..., N, \tag{4.17}$$

where Y_i is the output in terms of units of good i, and L_i is the hours worked by representative consumer i. (So one-hour working produces one unit.) She sells all her products in the market. If the price in terms of dollars is P_i, she makes $P_i Y_i$ dollars, which becomes her *nominal* income.

But no one consumes money itself, so nominal income is not important. That is, we have to control for the *overall price level* P to make it her *real* income. There are several ways to measure P, but here let us use the following *geometric* mean of all prices:

$$P \equiv \sqrt[N]{P_1 \times ... \times P_N}. \tag{4.18}$$

(This type of index makes our discussion more convenient.) Similarly, let us also denote by Y the *aggregate output*, which is also defined as a geometric mean of all outputs:

$$Y \equiv \sqrt[N]{Y_1 \times ... \times Y_N}. \tag{4.19}$$

The *real* income is, as usual, the nominal income (P_iY_i) divided by the overall price level. The real income, or equivalently *consumption*, becomes

$$C_i = \frac{P_iY_i}{P}. \tag{4.20}$$

Assume that the utility function of representative consumer i is given by

$$U(C_i, L_i) = C_i - \frac{(L_i)^\gamma}{\gamma}, \quad \gamma > 1. \tag{4.21}$$

One way to interpret (4.21) is to consider $U(C_i, L_i)$ as a profit of firm i. That is, $C_i = \frac{P_iY_i}{P}$ is the revenue, and $\frac{(L_i)^\gamma}{\gamma}$ is the labor cost (which is convex). Representative consumer i maximizes her utility, given the price of good i (P_i) and overall price level (P). That is, we assume:

Assumption for Model 1: *Each consumer i perfectly observes not only the price of good i (P_i), but also the overall price level (P).*

From (4.17), (4.20) and (4.21), she maximizes

$$\frac{P_iL_i}{P} - \frac{(L_i)^\gamma}{\gamma} \tag{4.22}$$

by choosing L_i. The first-order condition is

$$L_i = \left(\frac{P_i}{P}\right)^{\frac{1}{\gamma-1}}.$$

From (4.17), this equation also gives the output of industry i. That is,

$$Y_i = L_i = \left(\frac{P_i}{P}\right)^{\frac{1}{\gamma-1}}. \tag{4.23}$$

Since $\gamma > 1$, the power $\frac{1}{\gamma-1}$ is positive, which means that the labor input (L_i) or output (Y_i) increases in the *relative price of good i to the overall price level (P_i/P)*. The interpretation is clear: *As the good you produce becomes more expensive (relative to other goods), you will produce more units*. It is convenient to take logs on each side of (4.23) to have

$$y_i^S = \frac{1}{\gamma - 1}(p_i - p), \tag{4.24}$$

where $y_i^S \equiv \log(Y_i)$ and lower-case variables denote logs (e.g., $p_i \equiv \log(P_i)$). (4.24) describes the *supply* of good i given p_i and p.

We are looking for an equilibrium, so we also need an equation describing the *demand* for good i. We will simply assume that the demand is given by

$$y_i^D = y + z_i - \alpha(p_i - p), \quad \text{where } \alpha > 0. \tag{4.25}$$

Here, y is a constant. z_i represents preferences for good i, taking a higher value as it becomes more popular, satisfying $\sum_{i=1}^N z_i/N = 0$. And the final term $(-\alpha(p_i - p))$ implies that as good i becomes relatively more expensive (i.e., as $p_i - p$ increases), the quantity demanded decreases. If we accept (4.25) and take average over all goods, then we have

$$\frac{\sum_{i=1}^N y_i^D}{N} = y + \frac{\sum_{i=1}^N z_i}{N} - \alpha\left(\frac{\sum_{i=1}^N p_i}{N} - p\right).$$

But we know the second term in the right-hand side becomes zero by assumption. Furthermore, (4.18) implies

$$\log P = \frac{1}{N}(\log P_1 + ... + \log P_N),$$

that is,

$$\frac{\sum_{i=1}^N p_i}{N} = p, \tag{4.26}$$

so the third term also becomes zero. Hence, we finally have

$$\frac{\sum_{i=1}^N y_i^D}{N} = y. \tag{4.27}$$

So just as p in (4.26) is the log of overall price level, the constant term (y) of (4.25) is, in fact, the log of *aggregate demand*. ((4.19) introduces a similar definition for the aggregate output. In the equilibrium, aggregate demand equals aggregate supply which equals aggregate output, so they are all the same.)

In the equilibrium, demand equals supply, so from (4.24) and (4.25), we have

$$\frac{1}{\gamma-1}(p_i - p) = y + z_i - \alpha(p_i - p), \tag{4.28}$$

or equivalently,

$$\left(\frac{1+\alpha\gamma-\alpha}{\gamma-1}\right)(p_i - p) = y + z_i,$$

or equivalently,

$$p_i = \left(\frac{\gamma-1}{1+\alpha\gamma-\alpha}\right)(y + z_i) + p. \tag{4.29}$$

This equation determines how the (log) price of good i is related to the taste (z_i), (log) aggregate output (y), and (log) overall price level (p).

Our goal is to relate the total output (Y) to the price level (P) to see how changes in P affects Y. Averaging (4.29) over all $i = 1, ..., N$, we have

$$p = \left(\frac{\gamma-1}{1+\alpha\gamma-\alpha}\right)y + p,$$

by $p = \frac{1}{N}\sum_{i=1}^{N} p_i$ (which we showed!) and $\sum_{i=1}^{N} z_i/N = 0$ (by assumption). Hence,

$$\left(\frac{\gamma-1}{1+\alpha\gamma-\alpha}\right)y = 0,$$

but since $\frac{\gamma-1}{1+\alpha\gamma-\alpha} \neq 0$ from $\gamma > 1$, we have $y = 0$, which implies

$$Y = 1. \tag{4.30}$$

So the aggregate output is constant at 1, and is not affected by the overall price level, P. From (4.17), units of outputs are equal to units of labor inputs. If we define some sort of employment index by

$$L = \sqrt[N]{L_1 \times ... \times L_N} \tag{4.31}$$

just as (4.18) and (4.19), then (4.30) implies $L = 1$. So *the (un)employment level is not affected by the overall price level, P.*

Of course, the overall price level is determined by money supply controlled by the central bank. We have $MV = PY$. We assume $V = 1$, and we know from (4.30) that $Y = 1$. This means $M = P$. So a 10% increase in money supply

(M) only increases the price level (P) by 10%, but does not change any "real" variables such as the aggregate output or (un)employment index. This result is called **monetary neutrality**.

Result 4: *If the information for the overall price level is perfect, an increase in money supply affects the overall price level but not the aggregate output or (un)employment rate.*

[Show figures from Lucas (1995).] This result appears to be supported by many pieces of empirical evidence. If we look at many annual observations of the United States, inflation rates are positively related to money growth rates, but are *not* closely related to real GDP growth rates or unemployment rates.

However, it seems that some "subsets" of data (for example, 5 or 6 consecutive annual observations) sometimes reveal *negative* relationships between inflation rates and unemployment rates. This implies that *in the short run, contrary to our predictions, money growth may increase real outputs and decrease unemployment*. How can we explain the **Phillips curve** – the curve relating inflation rates to unemployment rates – which is sometimes downward-sloping in the short run?

4.2.2 Model 2: Imperfect Information on Price Level

A substantial number of economists are willing to accept the argument that money growth has no *long-run* effects. But *short-run* effects of money growth are expected to exist for various reasons. The following model, known as **Lucas island model**, provides theoretical explanation on these short-run effects.

The environment now is slightly different from Model 1. Assume different industries (i.e., different representative consumers) live in different "islands." A representative consumer still knows the price of the good she produces, but does not exchange information with others. So she only has "imperfect" information on the overall price level (P). That is, representative consumers only have their own *beliefs* (which are not necessarily identical to one another) on the overall price level.

To make the setup simple, we will assume that *all* representative consumers believe that the price level would *surely* be P^e, which is constant.

Assumption for Model 2: *Each consumer i observes only the price of good i (P_i), but not the overall price level (P). All consumers have a common forecast of the overall price level (P^e).*

Under this assumption, consumer i maximizes the following objective function (replacing (4.22)), by choosing L_i:

$$\frac{P_i L_i}{P^e} - \frac{(L_i)^\gamma}{\gamma}.$$

A new first-order condition (replacing (4.24)) becomes

$$y_i^S = \frac{1}{\gamma - 1}(p_i - p^e), \tag{4.32}$$

which describes the supply of good i.

The interpretation of this equation is important. Suppose that the overall price level increases, and because of this, the (nominal) price of good i also increases. Although this is simply a nominal increase, the representative consumer i may mistakenly interpret this as an increase in the *real* price (P_i/P). This misperception increases her production. *If all producers act like this, there will be an increase in aggregate output.*

On the other hand, assume that the demand function (4.25) is still valid. This implies that the equilibrium is made by equating y_i^D of (4.25) (i.e., demand) and y_i^S of (4.32) (i.e., supply), that is,

$$\frac{1}{\gamma - 1}(p_i - p^e) = y + z_i - \alpha(p_i - p),$$

(replacing (4.28)). Hence we have

$$\left(\frac{1}{\gamma - 1} + \alpha\right) p_i = y + z_i + \alpha p + \frac{1}{\gamma - 1} p^e.$$

Taking average over i, we have

$$\left(\frac{1}{\gamma - 1} + \alpha\right) p = y + \alpha p + \frac{1}{\gamma - 1} p^e.$$

(See the counterpart algebra in the previous subsection.) Or equivalently,

$$y = \frac{1}{\gamma - 1}(p - p^e). \tag{4.33}$$

This is the result, and now y (log aggregate output) *is* related to p (log overall price level).

The interpretation of (4.33) is as follows. If the forecast on overall price level is perfectly correct, i.e., $P = P^e$ or $p = p^e$, (4.33) implies $y = 0$. So even though the overall price level changes due to some change in monetary policy, the output is *not* affected as long as it is *anticipated*. The monetary policy is valid only when it is *unanticipated*. If the actual price level becomes higher [lower] than the forecasted level (i.e., $P > P^e$ or $p > p^e$), then (4.33) implies that output will increase [decrease].

We have similar results for the employment level. If we define $l \equiv \log(L)$ where L is defined in (4.31), then (4.33) also implies

$$l = \frac{1}{\gamma - 1}(p - p^e). \tag{4.34}$$

So unanticipated changes in the price level will affect (un)employment level of the economy. [Draw figures similar to Phillips curves.]

(4.33) also implies

$$Y = \left(\frac{P}{P^e}\right)^{\frac{1}{\gamma-1}}.$$

Inserting this into $MV = PY$ with $V = 1$,

$$M = P\left(\frac{P}{P^e}\right)^{\frac{1}{\gamma-1}},$$

or equivalently,

$$P = M^{\frac{\gamma-1}{\gamma}}(P^e)^{\frac{1}{\gamma}}.$$

Once the forecast (P^e) is given, the price level (P) still increases in the money supply (M) since $\frac{\gamma-1}{\gamma} > 0$. In particular, if $P = P^e$, we will have $P = M$.

Result 5: *An increase in money supply affects the overall price level. But only unanticipated movements in this overall price level affects real output or (un)employment. That is, if the overall price level rises more than anticipated, then the real output increases and the unemployment level decreases.*

If this result is correct, the Phillips curve, “sometimes” downward-sloping in the short run, can be explained. (4.34) can be written as

$$l_t = \frac{1}{\gamma - 1}(p_t - p_t^e)$$

by introducing some time subscript t, or equivalently,

$$l_t = \frac{1}{\gamma - 1}\left[\underbrace{(p_t - p_{t-1})}_{\pi_t} - \underbrace{(p_t^e - p_{t-1})}_{\pi_t^e}\right].$$

Log differences become the growth rates, so $p_t - p_{t-1}$ and $p_t^e - p_{t-1}$ are actual and anticipated inflation rates, denoted by π_t and π_t^e here. That is,

$$l_t = \frac{1}{\gamma - 1}\left(\pi_t - \pi_t^e\right). \tag{4.35}$$

So if π_t^e stays at a similar level for some period, then l_t (log employment) and π_t (inflation rate) are positively correlated, so we can say that unemployment and inflation are negatively related. This justifies a possible negative relationship between inflation rate and unemployment rate. Of course, if consumers change their beliefs, this negative relationship may not necessarily hold. That is, in the long run, the downward-sloping Phillips curve is not expected to be observed.

4.2.3 Discussion

The growth of money supply (M) is closely related to changes in price level (P), both theoretically and empirically. But it is more complicated to relate money growth or inflation rate (nominal term) to real GDP growth or unemployment rate (real term). Although a clear, long-term relationship between these variables does not seem to be observed (which implies monetary superneutrality), we cannot completely rule out the possibility of some sort of negative relationship in the short run.

One attempt to explain this is the island model that we have just discussed. Producers only have imperfect information on the overall price level, so they may mistakenly perceive nominal increases [decreases] in prices of the goods they produce as real ones. If this happens, producers produce more [less] units, increasing the aggregate output.

Although we do not cover in this lecture, another attempt to explain the effects of money growth on output and (un)employment is the **sticky-price model**. Prices of various products in the real world do not necessarily change whenever the overall price level moves. Perhaps it is costly to change the price tags every day. (This cost is called the **menu cost**.) Also in some cases, long-term contracts are made with fixed prices. If these are important, price changes may find their ways to affect real variables.

True or False? —

1. A higher level of money growth increases the real output and decreases the unemployment rate.

2. Because inflation rate and unemployment rate are negatively correlated, high inflation and high unemployment cannot coincide. (This is called **stagnation***.)*

Exercises

1. *(Seigniorage and Inflation)* Consider an economy which is identical to the one discussed in Section 4.1, except that the government is *not* allowed to borrow at all at any period. (Thus, in this economy, the government expenditure should be financed either by taxation or by seigniorage.)

i) Population grows at a constant rate, $n > 0$. That is, $N_t/N_{t-1} = 1 + n$, for all $t = 0, 1, ...$, where N_t denotes the population size.

ii) The real output per capita, y_t, is constant and equal to 1. That is, $y_t = 1$ for all $t = 0, 1, ...$

iii) The real government debt is always zero. That is, $B_t = 0$ for all $t = 0, 1, ...$

iv) There are positive deficits at all periods (i.e., $G_t - T_t > 0$). In particular, assume $(G_t - T_t)/N_t = d > 0$, for all $t = 0, 1, ...$, where G_t is the real government consumption and T_t is the real tax revenue.

v) The quantity theory of money holds with a constant velocity, $V_t = 1$. That is, $P_t Y_t = M_t$, for all $t = 0, 1, ...$, where P_t is the price level, Y_t is the real output of this economy, and M_t is the money stock.

vi) Seigniorage should be enough to satisfy the government budget constraint.

Answer the following questions. (You may assume period-(-1) values for all variables are given.)

(a) Write down the government budget constraint in this economy and explain what each term means.

(b) Describe the evolution of M_t given M_{t-1} and d. What is the money growth rate (i.e., $M_t/M_{t-1} - 1$) given d? How is this money growth rate related to d? What is your economic explanation about this relationship?

(c) Show how M_t evolves over time. (That is, draw a graph with time t on x-axis and M_t on y-axis.) Feel free to assume M_{-1} is given and start with time 0. If d becomes smaller, what happens to the evolution of M_t? What if d becomes larger? Explain why in plain words.

(d) Find an explicit expression for the inflation rate, $P_t/P_{t-1} - 1$. Suppose the population growth rate is $n = 1\%$. For what value of d is the inflation rate zero? Is it possible that the government enjoy seigniorage even though the inflation rate stays at zero? Why or why not? Explain in plain words. (If you want to, quote the quantity theory of money.)

2. *(Seigniorage and Inflation)* Consider an economy which is identical to the one discussed in Section 4.1:

i) Population grows at a constant rate, $n > 0$. That is, $N_t/N_{t-1} = 1 + n$, for all $t = 0, 1, ...$, where N_t denotes the population size.

ii) The real output per capita, y_t, is constant and equal to 1. That is, $y_t = 1$ for all $t = 0, 1, ...$

iii) The debt-to-GDP ratio cannot exceed a given level, $\bar{b}$. That is, $b_t \equiv B_t/Y_t \leq \bar{b}$ for all $t = 0, 1, ...$, where B_t is the real government debt and Y_t is the real output of this economy.

iv) There are positive deficits at all periods (i.e., $G_t - T_t > 0$). In particular, assume $(G_t - T_t)/N_t = d > 0$, for all $t = 0, 1, ...$, where G_t is the real government consumption, T_t is the real tax revenue, and d is a given constant.

v) The quantity theory of money holds with a constant velocity, $V_t = 1$. That is, $P_t Y_t = M_t$, for all $t = 0, 1, ...$, where P_t is the price level and M_t is the money stock at the end of t.

vi) The central bank keeps the money growth rate constant at θ whenever possible (i.e., before b_t reaches $\bar{b}$). That is, $M_t/M_{t-1} = 1+\theta$, for all $t = 0, 1, ...$, whenever possible.

vii) The government budget constraint is given by

$$G_t + (1 + r)B_{t-1} = T_t + B_t + \frac{M_t - M_{t-1}}{P_t},$$

where r is the real interest rate.

Based on this model, we derived:

$$b_t = \frac{1 + r}{1 + n} b_{t-1} + d - \left(1 - \frac{M_{t-1}}{M_t}\right),$$

for all $t = 0, 1, ...$ Throughout this question, assume $b_{-1} = 0$ is given. Further, assume $r > n$ unless otherwise noted.

Is the following statement TRUE or FALSE? Fully justify your answer by elaborating some of given equations. Provide intuitions whenever possible.

(a) If the government deficit is always zero (i.e., if $d = 0$), the central bank is able to keep the money growth rate at zero forever.

(b) If the government deficit is always zero (i.e., if $d = 0$), the central bank is able to keep the money growth rate at zero forever, and in this case, the inflation rate is always zero.

(c) Even though the government always has positive deficit (i.e., $d > 0$), it is possible that the government debt stays at zero forever by choosing some monetary policy.

(d) If the GDP growth rate is higher than the interest rate ($r < n$), then even though the government deficit is high relative to the level of money growth rate (i.e., $d - \frac{\theta}{1+\theta} > 0$), it is possible that the a catastrophe date never arrives (i.e., b_t does not reach $\bar{b}$ forever). (Hint: $\sum_{t=0}^{\infty} \alpha^t = 1/(1-\alpha)$ for $0 < \alpha < 1$.)

(e) The condition $d - \frac{\theta}{1+\theta} > 0$ implies that the government deficit is higher than seigniorage.

Chapter 5
Monetary Policy (II)

This chapter introduces theoretical approaches to find the optimal monetary policy.

5.1 Inflation, Unemployment and Optimal Monetary Policy ("Expectations-Augmented Phillips Curve")

Question: *Knowing the Relationship between Inflation and Unemployment, How Should the Central Bank Design the Monetary Policy? (Reference: Chapter 19 of DLS.)*

In a previous chapter, we attempted to find the optimal *fiscal* policy assuming that the government minimizes the present value of deadweight losses. In this section, our goal is to find the optimal *monetary* policy, where the central bank minimizes some sort of "losses" caused by inflation or unemployment. In particular, we will assume that the sum of squares of inflation rate and unemployment rate ($u^2 + \pi^2$,) is the objective to be minimized.

We start from (4.35). This equation argues that the employment is positively related to the actual inflation rate minus the expected. Let us write this as

$$u = u^* - b(\pi - \pi^e), \quad \text{where } b > 0. \tag{5.1}$$

Here, u is the unemployment rate (in terms of %), u^* is some constant, π is the *actual* inflation rate (%), and π^e is the *anticipated* inflation rate (%). This equation, often called the **expectations-augmented Phillips curve**, means that the unemployment rate is negatively related to the actual inflation rate minus the expected. Notice that if $\pi = \pi^e$, we have $u = u^*$, so the constant term (u^*) in (5.1) is the unemployment rate in which the actual inflation is perfectly anticipated. This level is called the **natural unemployment rate**, which is assumed to be constant in this setup.

The central bank is assumed to (equally) dislike unemployment and inflation. In particular, it minimizes

$$u^2 + \pi^2.$$

With (5.1), the central bank's problem can be written as

$$\min_{\pi} \left([u^* - b(\pi - \pi^e)]^2 + \pi^2 \right). \tag{5.2}$$

We assume that the central bank can freely choose the level of inflation rate (for example, by printing money). The optimal monetary policy is to set the inflation rate π which solves (5.2). Different assumptions on the anticipated inflation π^e yield different results.

Case 1. Fixed Expectations: First, assume π^e is fixed and observable. So π^e is a given constant, which is known to the central bank. The first-order condition to (5.2) is

$$-2b[u^* - b(\pi - \pi^e)] + 2\pi = 0,$$

or equivalently,

$$\pi = b[u^* + b(\pi^e - \pi)],$$

so the optimal inflation rate given π^e, which is denoted by $\pi^*(\pi^e)$, satisfies

$$\pi^*(\pi^e) = \underbrace{\frac{b}{1+b^2}u^*}_{\text{positive intercept}} + \underbrace{\frac{b^2}{1+b^2}}_{\text{positive slope, smaller than 1}} \times \pi^e. \tag{5.3}$$

[Draw a graph.] Notice that the slope ($\frac{b^2}{1+b^2}$) is positive. So if the anticipated inflation is high [low], the central bank should keep the actual inflation high [low]. Also notice that the intercept is positive but the slope is lower than 1. This implies that when the anticipated rate of inflation is relatively low [high], the optimal rate of inflation should be higher [lower] than the expectation.

Case 2. Nash Equilibrium: But this solution is based on "fixed" expectations. Consumers may know that the central bank's target is $\pi^*(\pi^e)$ in (5.3). This information will change their anticipated rate of inflation. If so, $\pi^*(\pi^e)$ is no longer optimal because π^e itself is changed. [Use the graph to make this clear.] This environment becomes a **game** in which the central bank chooses π^* and consumers choose π^e. The objective of central bank is to solve (5.2), and that of consumers is to anticipate the inflation rate as accurately as possible. As in a usual perfect-information game, consumers in this game know about the central bank's problem. Also, the central bank knows that consumers know about the central bank's problem. Further, consumers know that the central bank knows that consumers know about the central bank's problem. ... (This continues forever.) Similarly, the central bank knows about the problem of consumers, consumers know that the central bank knows about the problem of consumers, etc.

Let us solve this game. In a **Nash equilibrium**, players maximize their payoffs (or minimize their costs) given the strategies of others. It is clear that the central bank will choose $\pi^*(\pi^e)$ in (5.3) once the consumer's strategy π^e is given. Consumers will set π^e to be $\pi^*(\pi^e)$ once the central bank's strategy is given. That is, we should have

$$\pi^*(\pi^e) = \pi^e$$

in a Nash equilibrium. Hence, (5.3) implies

$$\pi^* = \frac{b}{1+b^2}u^* + \frac{b^2}{1+b^2}\pi^*,$$

or equivalently,

$$\pi^* = bu^*,$$

which means *the optimal monetary policy is to keep the inflation rate constant at* bu^*%. In this Nash equilibrium, both expected and actual inflation rates become this level at π^*. So (5.1) implies that the unemployment rate becomes

$$\begin{aligned} u &= u^* - b(\pi^* - \pi^*) \\ &= u^* \end{aligned}$$

So the unemployment rate will also be constant at the natural level.

Case 3. Ramsey Equilibrium: Can't we do better? We can also imagine the following consequence, called the **Ramsey equilibrium**. Let the central bank announce the **target inflation rate** to the public. (And assume that this announcement *is* credible.) Suppose that the announced target is π^{**}. Also suppose that consumers trust the central bank, so the anticipation of consumers is at the same level, π^{**}. So (5.1) implies that

$$\begin{aligned} u &= u^* - b(\pi^{**} - \pi^{**}) \\ &= u^* \end{aligned}$$

Hence, the unemployment rate is not affected by the target inflation rate as long as it is credible. Recall that the government minimizes

$$u^2 + \pi^2 = (\underbrace{u^*}_{\text{constant}})^2 + (\underbrace{\pi^{**}}_{\text{target inflation rate}})^2,$$

so the solution is obviously $\pi^{**} = 0$. To summarize, *in the Ramsey equilibrium, the inflation rate is zero, (that is, the central bank announces 0% inflation as a target, consumers trust the central bank, and the central bank actually sets 0% inflation as promised,) and the economy has the natural unemployment rate.*

Of course, to obtain this result, the central bank needs to credibly commit to the target. But notice that the central bank may choose to *deviate*. That is, if consumers actually anticipate 0% inflation, (5.3) implies that the optimal inflation rate is now $\frac{b}{1+b^2}u^*$%, so the central bank may break its promise to increase the inflation rate. Because of such a worry, consumers will not trust the central bank's 0% target, so this Ramsey equilibrium (which requires consumers to accept the target) will be hard to be attained again. This is similar to Prisoner's dilemma in which both players can be better off if both choose to deny.

The credibility of monetary policy is therefore important. It is better for the central bank to keep the inflation as low as possible and to announce this target to the public to affect the anticipation. To do this, the central bank should show its credibility by keeping promises for a long time. Perhaps, this is one reason why the Federal Reserve tries its best to convince the people that it is very seriously fighting against the inflation.

True or False? —

1. If we look at annual observations of any four or five consecutive years, the inflation rate should be inversely related to the unemployment rate.

2. Suppose the price level has been constant for a long time until year 2006. A news is released, at the beginning of 2006, that the central bank will increase the money supply in that year. If all people expect that the price level would increase by 1% (in 2006), and if it actually increases by 2% (in 2006), then the unemployment rate will increase.

3. Page 235 of DLS says: "Paul Volcker [in early 1980s] arrived as Chairman of Federal Reserve Board at a time of high inflation and high unemployment. He announced that there would be low inflation in the future. The private sector did not adjust its expectations, but Volcker followed through on his promise." Then, the unemployment would have stayed above the natural rate for a while.

4. In the optimal monetary policy problem minimizing the sum of squares of inflation rate (%) and unemployment rate (%), a Nash equilibrium gives the optimal result.

5. Assume an expecations-augmented Phillips curve, $u_t = u^* - b(\pi_t - \pi_t^e)$*, where* u_t *is the unemployment rate at period* t*,* u^* *is a positive constant equals to natural unemployment rate,* b *is a positive constant,* π_t *is the actual inflation rate at period* t*, and* π_t^e *is the anticipated inflation rate at period* t*. Suppose that the Federal Reserve wants to minimize the discounted value of sum of squares of inflation rate and unemployment rate (i.e.,* $\sum_{t=0}^{\infty} \beta^t(\pi_t^2 + u_t^2)$*) by choosing the inflation rates. Further, suppose that the inflation rate anticipated by consumers is given by 0% in period 0. Then, in period 0, the Federal Reserve would choose an inflation rate which is strictly positive if the anticipated inflation rates in periods 1, 2, ... depend on the inflation rates of previous periods (periods 0, 1, ..., respectively). [Correct intuitions are enough. You don't have to be formal in your solution. If you want to be formal and if you need, you are allowed to make any reasonable assumptions.]*

6. Because the fiscal and monetary policies should harmonize, the central bank should be under a perfect control of the government.

5.2 Interest Rate and Optimal Monetary Policy ("Friedman Rule")

> **Question:** *How Should the Central Bank Design the Monetary Policy to Minimize the Cost of Holding Money? And What are the Tools of Monetary Policy in Practice? (References: Sections 17.1 and 19.4 of DLS and the website of Board of Governors of the Federal Reserve System (http://www.federalreserve.gov).)*

We have assumed that the central bank simply "hates" inflation. But why? In this section, we continue to think about the optimal monetary policy, but with some theory about why inflation is bad. Savings provide interests, but holding money doesn't. We should hold some amount of money for daily purchases, which is costly because interests are not paid. Can the central bank do something to minimize the cost of money holding? This is the main question of the section.

First of all, we need a story relating (nominal) interest rate to (anticipated) inflation. The following **Fisher formula** provides this relationship:

$$R \approx r + \pi^e, \tag{5.4}$$

where R is the *nominal* interest rate, r is the *real* interest rate, and π^e is the anticipated inflation rate. This formula means the following: If someone borrows one unit of consumption good in this period, he/she should pay back $(1 + r)$ (for example, 1.02) units in the next period. But suppose the inflation is expected to be π^e (for example, 0.03 or 3%). This means one unit of consumption good, that is worth \$1 in this period, will be worth $\$(1 + \pi^e)$ in the next period. So in terms of dollars, if someone borrows \$1 (which is worth one unit of consumption good) in this period, he/she should pay back $\$(1 + r)(1 + \pi^e)$ dollars (which is worth $(1 + r)$ units of consumption goods) in the next period. This implies that the (net) nominal interest rate for money should be $r + \pi^e + r\pi^e$. If $r\pi^e$ (for example, 0.02×0.03=0.0006) is negligible, (5.4) is a reasonable approximation.

In particular, assume that (5.4) holds with an equality (for simplicity) and that the inflation rate is correctly forecasted. That is,

$$R = r + \pi. \tag{5.5}$$

Notice that under this assumption (on correct forecast of inflation), the unemployment always stays at its natural level. This enables us to forget about the labor market and to concentrate on minimizing the costs arising (directly) from inflation. We further assume that r in (5.5) is determined in the financial market and is not controlled by the central bank. The central bank can influence the nominal interest rate (R) by affecting the inflation rate (π) only.

5.2.1 Model Description

Now let us describe the model. A consumer has all her asset in the bank, which pays the nominal interest of R at each period. She goes to the bank x times per period to withdraw money for daily transactions. Assume that there is a fixed (constant) cost of μ dollars to withdraw any amount of money. (This may include time cost, transportation cost and transcation cost.) That is, withdrawing x times costs

$$\mu x \tag{5.6}$$

dollars. Also, there are opportunity costs from holding money. Assume this consumer withdraws c dollars in one period. (For simplicity, assume c is given.)

She goes to the bank x times in one period, so she withdraws c/x dollars every time. So she starts living with c/x dollars right after each withdrawal, constantly spends money until her money holding becomes 0, when she withdraws another c/x dollars. This means that she will, on average, hold $c/2x$ dollars and *lose* the interests paid to this amount. That is, she loses

$$\frac{c}{2x}R \tag{5.7}$$

dollars. Therefore, her total cost of holding money becomes

$$\mu x + \frac{c}{2x}R \tag{5.8}$$

dollars. The consumer chooses x to minimize this. There is a trade-off. If x is small (i.e., if she rarely goes to the bank), then cost of withdrawal, (5.6), is small, but opportunity cost of holding money, (5.7), is large.

5.2.2 Solution and Discussion

The first-order condition in this minimization problem is

$$\mu - \frac{cR}{2x^2} = 0,$$

or equivalently,

$$x = \sqrt{\frac{cR}{2\mu}}.$$

In this case, (5.8) implies that the total cost of this consumer from holding money becomes

$$\mu\sqrt{\frac{cR}{2\mu}} + \frac{c}{2\sqrt{\frac{cR}{2\mu}}}R = \sqrt{2cR\mu}, \tag{5.9}$$

given c (amount of dollars required for consumption), μ (unit cost of withdrawal) and R (nominal interest rate).

The central bank, able to control R, should try to minimize this cost, (5.9). Obviously, the solution to minimize this cost is

$$R = 0.$$

In this solution, (5.5) implies that the optimal inflation rate becomes

$$\pi = -r.$$

This result – that the inflation should be negative – is called the **Friedman rule**. Under the Friedman rule, the opportunity cost of holding money, (5.7), becomes zero, which means consumers hold all assets as money in order to minimize costs from withdrawal, (5.6).

Perhaps this result is not really practical for an actual monetary policy. When the nominal interest rate is zero, no one will save his/her money at banks, so financial intermediaries do not exist anymore. Also, deflation is harmful in production because simply holding money is sometimes better than investing. However, this result is interesting at least theoretically. The model implies that there are costs to consumers when they have inflation (and in fact, even mild deflation). Consumers are always losing something because they should go to the bank and because holding money does not pay interests. These costs become zero only under the Friedman rule.

5.2.3 Monetary Policy in Practice

In this subsection, we discuss three main tools of monetary policy in practice.

1. Reserve Requirements: Banks in the United States should hold a given fraction of their deposits, as cash in their vaults or as deposits with Federal Reserve Banks with no interests. (Banks may hold even more reserves than requirements to restock ATMs, to clear overnight checks, etc.) Changing the required fraction of these **reserve requirements** affects the money stock (M) in the following way.

Let us consider a hypothetical economy in which consumers always just save their money in banks. Assume that the required fraction is set to 10% by the Fed. The Fed printed \$100 for circulations. Consumers save this money in banks. The banks need to reserve 10%, or \$10, while \$90 are lent out to other consumers. These consumers, again, save this amount (\$90) at banks. Banks again reserve \$9, and lend out \$81 to the people. This continues forever. So the printed amount of dollars, \$100, now become the money stock of

$$100 + 100 \times 0.9 + 100 \times (0.9)^2 + ... = 100 \times \frac{1}{1 - 0.9} = 1000$$

by $\sum_{t=0}^{\infty} \alpha^t = 1/(1 - \alpha)$ for $0 < \alpha < 1$. As the reserve requirements become higher [lower] (i.e., as the policy becomes contractionary [expansionary]), the money supply becomes smaller [larger] because more [less] money should be reserved.

2. Open-Market Operations: Banks, in order to satisfy reserve requirements, may borrow or lend among themselves at the close of business each day. (That is, banks with reserve surpluses will lend money to others.) The interest rate used in these transactions is called the **federal funds rate**. The Fed can affect this federal funds rate by purchasing national securities from banks. This procedure is called the **open-market operation**. If the Fed purchases them, banks come to have more reserve in response with the Fed. This will decrease the federal funds rate because banks, in general, have more federal funds. At the same time, it will increase the money supply because the additional amount of federal funds can be traded for money. This type of policy – the Fed's purchasing national securities – is expansionary in the sense that it decreases the interest rate and increases the money supply. (On the other hand, selling national securities is contractionary.)

3. Discount Rates: "The **discount rate** is the interest rate charged to commercial banks and other depository institutions on loans they receive from their regional Federal Reserve Bank's lending facility – the **discount window**." (Federal Reserve website) The Fed may increase (which is contractionary) or decrease (which is expansionary) this rate to affect market interest rates.

True or False? —

1. The Friedman rule is obtained by minimizing the costs of inflation and unemployment.

2. A low rate of inflation is not costly at all.

3. Consider the (informal) setup of the Friedman rule, in which a consumer minimizes the sum of (i) the cost of withdrawals and (ii) the opportunity cost from losing interests, by choosing number of withdrawals. Even though the consumer confronts deflation of 1% (where the price level of the next period is 99% of that of this period), the sum of two costs may be positive.

4. Let us consider the setup that we considered when discussing the Friedman rule. A consumer minimizes a total cost, $\mu x + \frac{c}{2x}R$, which is the sum of withdrawal costs (μx) and opportunity costs from holding cash ($cR/2x$), by choosing the number of costly withdrawals (x). Here, μ is the cost of each withdrawal (in dollars), x is the number of costly withdrawals, c is the consumption (in dollars) *in one period, which is treated as given, and R is the nominal interest rate. As the nominal interest rate (R) rises by 1%, the number of withdrawals (x^*) optimally chosen by the consumer decreases by 0.5%.*

5. C.A.E. Goodhart, an economic advisor to the Bank of England for many years, suggests having a lottery based on cash serial numbers. If we introduce this lottery, the Friedman rule does not necessarily imply that the nominal interest rate should be zero.

Exercises

1. *(Inflation and Unemployment)* Suppose that the following expectations-augmented Phillips curve holds:

$$u_t = u^* - (\pi_t - \pi_t^e),$$

for all $t = ..., -2, -1, 0, 1, 2, ...$ (So b in the lecture note is assumed to be 1.) Here, u_t is the unemployment rate (%), u^* is the *natural* unemployment rate (%), π_t is the *actual* inflation rate (%), and π_t^e is the *anticipated* inflation rate (%). Assume all these four are non-negative.

We are at time 0, and the inflation rate before time 0 (at times -1, -2, ...) was constant at 10% (i.e., $\pi_{-1} = \pi_{-2} = ... = 10$). The central bank, able to fully control actual inflation rates, minimizes at time 0

$$\sum_{t=0}^{\infty}(u_t^2 + \pi_t^2),$$

by choosing $\pi_0, \pi_1, \pi_2, ...$ (There is no time discount factor (β) in this expression. That is, we take $\beta = 1$. We consider a general value of β in question (e).) Notice that the optimal inflation rate does not grow forever. If it does, the above expression will explode.

(a) Suppose that consumers have full information, so they always anticipate the correct level of inflation rate, i.e., $\pi_t^e = \pi_t$ for all $t = 0, 1, 2, ...$ Obtain the optimal inflation rate. Obtain the unemployment rate. How do they evolve over time?

(b) Suppose that consumers have "adapted expectations" so that $\pi_t^e = \pi_{t-1}$.

i) Show that the first-order condition satisfies

$$\pi_{t+1} - 3\pi_t + \pi_{t-1} = 0$$

for all $t = 0, 1, 2, ...$

ii) "Guess" that the central bank sets $\pi_t = \alpha\pi_{t-1}$, for all $t = 0, 1, 2, ...$, for some constant $\alpha > 0$. This α cannot be higher than 1 if we follow the assumption in the question. (That is, π_t does not grow forever.) Obtain α.

iii) Describe how the optimal inflation rate and the unemployment rate move over time.

(c) Why do your answers to these two questions differ? In particular, why don't the central bank change the inflation rate more aggressively in Question 2? Explain in plain words.

(d) [OPTIONAL: The solution will not be provided.] Suppose that consumers are between the above two cases so that $\pi_t^e = (\pi_{t-1} + \pi_t)/2$. This means that consumers do not 100% correctly predict the inflation rate, but at the same time, they are not foolish enough to just adapt today's inflation rate to predict tomorrow's. Describe how the optimal inflation rate and the unemployment rate evolve over time. Is the solution to this question somewhere between those to the previous questions?

(e) [OPTIONAL: The solution will not be provided.] How should we change the answers to the previous questions if the central bank minimizes

$$\sum_{t=0}^{\infty} \beta^t (u_t^2 + \pi_t^2), \quad \text{for } 0 < \beta < 1$$

instead?

2. *(A Revised Baumol-Tobin Model for Friedman Rule)* In our discussion on the Friedman rule, we considered a model in which it is costly for a consumer to withdraw money. But withdrawal technologies have evolved, and now we have ATMs in many places from which we can withdraw money with almost no costs. Our goal in this problem is to understand how the model's predictions on cash holding change with this new environment.

Let us consider the original setup that we discussed once. A consumer minimizes a total cost, which is the sum of withdrawal costs (μx) and opportunity costs from holding cash ($cR/2x$):

$$\mu x + \frac{c}{2x} R,$$

by choosing the number of costly withdrawals (x). Here, μ is the cost of each withdrawal (in dollars), x is the number of costly withdrawals, c is the consumption (in dollars) in one period, which is treated as given, and R is the nominal interest rate.

(a) Obtain the first-order condition of the problem.

(b) As the nominal interest rate rises, does the number of costly withdrawals (x) also increase? Why? (Provide intuitions.) What is the interest-rate elasticity of number of withdrawals? (That is, when the interest rate rises by 1%, by what % does the number of withdrawals increase?)

(c) As the nominal interest rate rises, does the average cash holding (i.e., the average amount of cash that this consumer holds at any moment) also increase? Why? (Provide intuitions.) And what is the interest-rate elasticity of cash holding?

(d) This original setup needs to be changed reflecting the development of withdrawal technologies. As before, the consumer can still go to the bank to withdraw money with unit cost μ. In addition, she meets the ATM p times from which she can withdraw money without costs. The value of p is given exogenously (and p becomes higher as withdrawal technologies develop). In sum, this consumer's free withdrawals are limited to p times, while she makes costly withdrawals (with cost of μ dollars each) x times. Set up the problem of the consumer to minimize the total cost. Obtain the first-order condition. (Assume in each of costly or free withdrawals, she withdraw the same amount of money. And assume she can make free or costly withdrawal at any moment she wants. If you need, make some reasonable assumptions.) (Hint: She chooses $x \geq 0$, the number of times she withdraws costly.)

(e) Suppose a consumer's optimal solution is to choose $x = 0$ (that is, she does not make costly withdrawals at all) for some given parameter values. How many free withdrawals does she make? Why? (Provide intuitions.)

(f) For simplicity, assume that the entire development of withdrawal technologies is represented by increasing availability of ATMs. Hence, consumers without ATM cards do not enjoy any benefits from this development while those with ATM cards do. Let us consider an economy with many consumers who have identical parameters except p. That is, this economy consists of identical consumers but the number of free withdrawals allowed is different across consumers.

Is the following statement TRUE or FALSE? In each of these, you should explain the reason using both mathematical results and interpretations/intuitions.

i) An unpublished paper written by Fernando Alvarez and Francesco Lippi reports the following statistics based on the survey conducted in Italy in year 2004. Reported numbers are sample means across many households.

Table 1: Number of (free or costly) Withdrawals per Year

Consumers without ATM cards	Consumers with ATM cards
23.0	63.1

As shown in Table 1, the development of withdrawal technologies (i.e., the overall increase in individual p's) will increase the average number of (free or costly) withdrawals.

ii) The same paper also reports:

Table 2: Average Cash Holding to Daily Consumption (c) ratio

Consumers without ATM cards	Consumers with ATM cards
17.8	13.6

As shown in Table 2, the development of withdrawal technologies (i.e., the overall increase in individual p's) will decrease the "Average Cash Holding to Daily Consumption" ratio.

iii) Interest-rate elasticity of number of withdrawals for an average consumer will be lowered as p rises.

iv) Interest-rate elasticity of cash holding for an average consumer, in absolute value, will be lowered as p rises.