EconS 502: Macroeconomic Theory II, Spring 2010
Professor: S. Choi
Due: In Class, Tuesday, 2nd February

PROBLEM SET 3

Form a group of 1-4 students. A group can submit one solution.

1. Consider the following economy. The representative consumer solves

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt,\ \sigma > 0,\ \sigma \neq 1,$$

where $C(t)$ is the aggregate consumption. There is no population growth or productivity growth. The only source of economic growth is physical capital accumulation. So the constraints are

$$\begin{aligned} Y(t) &= B(1-\tau)K(t),\ 0 \le \tau < 1,\ B > 0, \\ Y(t) &= C(t) + I(t), \\ \dot{K}(t) &= I(t) - \delta K(t),\ 0 < \delta < 1, \end{aligned}$$

where $Y(t)$ is the aggregate output, B is the constant productivity, τ is a flat tax rate, $K(t)$ is the stock of physical capital, $I(t)$ is the aggregate investment, and δ is a depreciation rate.

(a) Set up the Hamiltonian function and obtain the first-order conditions.
Hint:
- $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} h(x(t), u(t)) dt$ s.t. $\dot{x}(t) = g(x(t), u(t))$, $x(0)$ given.
- $H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \lambda(t) g(x(t), u(t))$.
- First-order conditions: $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.

(b) Assume a balanced growth path in which all variables grow at constant rates (or stay at constant levels). Calibrate the model using the following observations. Assume $\sigma = 2$.
- $\tau = 36\%$: McGrattan and Prescott (Federal Reserve Bank of Minneapolis Quarterly Review, 2000), Table 2.

- $s \equiv I_t/Y_t = 0.20$: 1960-2005, NIPA Table 1.1.5.
- $g_Y = 2.2\%$: 1960-2005, NIPA Table 7.1. This is the growth of GDP per worker. Yes, $Y(t)$ is aggregate GDP, not GDP per worker. But when the model doesn't have population growth, it is more reasonable to use the growth of GDP per worker because we get the same result in that way as in a model with population growth.
- $\delta K/Y = 0.12$: 1960-2005, NIPA Table 1.1.

(c) If taxes on physical capital income are eliminated so that $\tau = 0$, what would be the new growth rate of $Y(t)$ on the new balanced growth path?

(d) What would be the new level of s on this new balanced growth path?

2. The tax rate on labor income or physical capital income is almost 40% in the United States. Suppose that you want to analyze what will happen to the level of and the growth rate of per-capita output when the income tax is completely eliminated. Assume that the government throws away all tax revenue to the ocean.

   Consider the following economy. The representative consumer solves

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L)^{1-\sigma}}{1-\sigma} L dt, \ \rho > 0, \ \sigma > 0, \ \sigma \neq 1,$$

   where $\rho > 0$ is a parameter, $C(t)$ is the aggregate consumption at period t, and $L > 0$ is the constant population. (There is no population growth.) The constraints are

$$\begin{aligned} Y(t) &= K(t)^{\alpha}(A(t)L)^{1-\alpha}, \ 0 < \alpha < 1, \\ Y(t) &= C(t) + I(t) + \tau Y(t), \ 0 \leq \tau < 1, \\ \dot{K}(t) &= I(t) - \delta K(t), \ 0 < \delta < 1, \\ A(t) &= A(0)e^{g_A t}, \ g_A \geq 0 \text{ and } A(0) \text{ given}, \end{aligned}$$

   where $Y(t)$ is the aggregate output, $K(t)$ is the aggregate physical capital stock, α is a parameter, $A(t)$ is the productivity exogenously growing at g_A, $I(t)$ is the aggregate investment, τ is a flat tax rate on total income, and δ is a depreciation rate. Normalize $L = 1$. Eliminating $Y(t)$ and $I(t)$, the four constraints above are re-written as the following two constaints:

$$\begin{aligned} \dot{K}(t) &= (1-\tau)K(t)^{\alpha}A(t)^{1-\alpha} - C(t) - \delta K(t), \ 0 < \delta < 1, \\ A(t) &= A(0)e^{g_A t}, \ g_A \geq 0 \text{ and } A(0) \text{ given}. \end{aligned}$$

To eliminate one more constraint, define

$$k(t) \equiv \frac{K(t)}{A(t)},\ c(t) \equiv \frac{C(t)}{A(t)},$$

and

$$\eta \equiv \rho - (1-\sigma) g_A.$$

Then, we can re-write the problem as

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t} \frac{(c(t))^{1-\sigma}}{1-\sigma} L dt$$

subject to

$$\dot{k}(t) = (1-\tau)\,(k(t))^{\alpha} - c(t) - (\delta + g_A) k(t).$$

(a) Obtain the first-order conditions.

Hint:

- $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t)) dt$ s.t. $\dot{x}(t) = g(x(t), u(t))$, $x(0)$ given.
- $H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \lambda(t) g(x(t), u(t))$.
- First-order conditions: $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.
  (Don't forget that our problem has η in place of ρ here!)

(b) Derive the Euler equation. (That is, eliminate $\lambda(t)$ in the system of two equations you derived in (a). Then express it with original notations such as $K(t)$, $C(t)$, $A(t)$ and ρ, eliminating $k(t)$, $c(t)$ and η.)

(c) Consider the economy on the balanced growth path, in which all variables grow at constant rates. Show that on this balanced growth path, $Y(t)$, $C(t)$ and $A(t)$ all grow at the same rate (i.e., $g_Y = g_C = g_A$). Be as explicit as possible.

(d) Suppose $\tau = 0.400$, $\alpha = 0.320$, $\delta = 0.050$, $\rho = 0.040$, $\sigma = 2$, and $g_Y = 0.022$. What is the (constant) level of $K(t)/Y(t)$ on the balanced growth path?

(e) Suppose that τ had been 40% up to year $t = 2009$, and all of a sudden, the government eliminates the tax so that $\tau = 0$ thereafter. All parameter values exogenously given (α, δ, ρ, σ and g_A) are the same as before. Suppose that after several years of "transition", the economy has reached a new balanced growth path. What would be the new (constant) level of $K(t)/Y(t)$ on this new balanced growth path?

(f) Now let us compare two scenarios. In Scenario 1, $\tau = 0.4$ is imposed forever. In Scenario 2, $\tau = 0.4$ had been believed to be imposed forever, but as in (e), all of a sudden, the government eliminates the tax in $t = 2009$ so that $\tau = 0$ thereafter. Suppose at least in $t = 2050$, Scenario 2's economy would reach a new balanced growth path, as described in (e). Obtain the ratio of $Y(2050)$ in Scenario 2 to $Y(2050)$ in Scenario 1. That is, how much more output does this economy produce in Scenario 2? (Hint: $Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\alpha/(1-\alpha)} A(t)$.)

(g) Draw a figure that shows your result as clearly as possible. On the x-axis is time t. On the y-axis is output ($Y(t)$). Draw Scenario 1's output path. Draw Scenario 2's output path in the same figure. (You don't have to draw the path on the "transition" before the new balanced growth path is reached.) Are growth rates different between these two scenarios? Then why might the government want to decrease the tax rate?