EconS 502: Macroeconomic Theory II
Professor: S. Choi
Wednesday, 18th February, 2009

MID-TERM EXAM 1
(75 minutes, 100 points)

1. [30 points, 15 points each] Are the following statements TRUE or FALSE? Provide your theoretical and/or empirical reasonings based on our class discussions. The credits will be solely based on your reasonings.

   (a) Each individual will determine his or her optimal level of schooling. There is no need for the government to intervene in education.

   (b) Assume that for the past 10 years, the real per-capita GDP of China has grown at 8% per year. For the next 10 years, it is likely to grow slower than 8% per year.

2. [70 points, 10 points each] The tax rate on labor income or physical capital income is about 40% in the United States. Suppose that you want to analyze what will happen to the level of and the growth rate of per-capita output when the income tax is completely eliminated. Assume that the government throws away all tax revenue to the ocean.

   Consider the following economy. A representative consumer solves

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{(C(t)/L)^{1-\sigma}}{1-\sigma}Ldt,\ \rho > 0, \sigma > 0,\ \sigma \neq 1,$$

   where ρ is a parameter, $C(t)$ is aggregate consumption at period t, and L is constant population. (There is no population growth.) The constraints are

$$\begin{aligned}
Y(t) &= K(t)^{\alpha}(A(t)L)^{1-\alpha},\ 0 < \alpha < 1,\\
Y(t) &= C(t) + I(t) + \tau Y(t),\ 0 \leq \tau < 1,\\
\dot{K}(t) &= I(t) - \delta K(t), 0 < \delta < 1,\\
A(t) &= A(0)e^{g_A t},\ g_A \geq 0 \text{ and } A(0) \text{ given},
\end{aligned}$$

   where $Y(t)$ is aggregate output, $K(t)$ is aggregate physical capital stock, α is a parameter, $A(t)$ is the productivity exogenously growing at g_A, $I(t)$ is aggregate investment, τ is a flat tax rate on total income, and δ is a depreciation rate. Normalize $L = 1$. Eliminating $Y(t)$ and $I(t)$, the four constraints above are re-written as the following two constaints:

$$\begin{aligned}
\dot{K}(t) &= (1-\tau)K(t)^{\alpha}A(t)^{1-\alpha} - C(t) - \delta K(t), 0 < \delta < 1,\\
A(t) &= A(0)e^{g_A t},\ g_A \geq 0 \text{ and } A(0) \text{ given}.
\end{aligned}$$

   To eliminate one more constraint, define

$$k(t) \equiv \frac{K(t)}{A(t)},\ c(t) \equiv \frac{C(t)}{A(t)},$$

and

$$\eta \equiv \rho - (1-\sigma)g_A.$$

Then, we can re-write the problem as

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t} \frac{(c(t))^{1-\sigma}}{1-\sigma} L dt$$

subject to

$$\dot{k}(t) = (1-\tau)(k(t))^{\alpha} - c(t) - (\delta + g_A)k(t).$$

(a) Obtain the first-order conditions.
Hint:

- $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t)) dt$ s.t. $\dot{x}(t) = g(x(t), u(t))$, $x(0)$ given.
- $H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \lambda(t) g(x(t), u(t))$.
- First-order conditions: $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.
(Don't forget that our question has η in place of ρ here!)

(b) Derive the Euler equation. (That is, eliminate $\lambda(t)$ in the system of two equations you derived in (a). Then express it with original notations such as $K(t)$, $C(t)$, $A(t)$ and ρ, eliminating $k(t)$, $c(t)$ and η.)

(c) Consider the economy on the balanced growth path, in which all variables grow at constant rates. Show that on this balanced growth path, $Y(t)$, $C(t)$ and $A(t)$ all grow at the same rate (i.e., $g_Y = g_C = g_A$). Be as explicit as possible.

(d) Suppose $\tau = 0.400$, $\alpha = 0.320$, $\delta = 0.050$, $\rho = 0.040$, $\sigma = 2$, and $g_Y = 0.022$. What is the (constant) level of $K(t)/Y(t)$ on the balanced growth path?

(e) Suppose that τ had been 40% upto year $t = 2009$, and all of a sudden, the government eliminates the tax so that $\tau = 0$ thereafter. All parameter values exogenously given (α, δ, ρ, σ and g_A) are the same as before. Suppose that after several years of "transition", the economy has reached a new balanced growth path. What would be the new (constant) level of $K(t)/Y(t)$ on this new balanced growth path?

(f) Now let us compare two scenarios. In Scenario 1, $\tau = 0.4$ is imposed forever. In Scenario 2, $\tau = 0.4$ had been believed to be imposed forever, but as in (e), all of a sudden, the government eliminates the tax in $t = 2009$ so that $\tau = 0$ thereafter. Suppose at least in $t = 2050$, Scenario 2's economy would reach a new balanced growth path (as described in (e)). Obtain the ratio of $Y(2050)$ in Scenario 2 to $Y(2050)$ in Scenario 1. That is, how much more output does this economy have in Scenario 2? (Hint: $Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\alpha/(1-\alpha)} A(t)$.)

(g) Draw a figure that shows your result as clearly as possible. On the x-axis is time t. On the y-axis is output ($Y(t)$). Draw Scenario 1's output path. Draw Scenario 2's output path. (You don't have to draw the path on the "transition" before the new balanced growth path is reached.) Are growth rates different between these two scenarios? Then why might the government want to decrease the tax rate?

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1. [30 points, 15 points each] Are the following statements TRUE or FALSE? Provide your theoretical and/or empirical reasonings based on our class discussions. The credits will be solely based on your reasonings.

   (a) Each individual will determine his or her optimal level of schooling. There is no need for the government to intervene in education.

   Answer: *FALSE. The government may want to subsidize the educational sector for the following two reasons. (i) The capital market is not perfect, so young people may not borrow enough to go to school. (ii) There are human capital externalities. Workers benefit from the human capital stocks owned by other workers. In individual decision of schooling, this social benefit is disregarded, so education is under-invested, which may call for government intervention.*

   (b) Assume that for the past 10 years, the real per-capita GDP of China has grown at 8% per year. For the next 10 years, it is likely to grow slower than 8% per year.

   Answer: *TRUE. China's growth as high as 10% per year can be viewed as an episode of "catch-up growth". As in Lucas (2009), China's growth can be modeled as* $\dot{Y}_{CHN}(t)/Y_{CHN}(t) = 0.02[Y_{USA}(t)/Y_{CHN}(t)]^{\theta}$. *So as* $Y_{CHN}(t)$ *becomes higher (compared to* $Y_{USA}(t)$ *which grows only at 2% per year), the growth rate will become lower.*

2. [70 points, 10 points each] The tax rate on labor income or physical capital income is about 40% in the United States. Suppose that you want to analyze what will happen to the level of and the growth rate of per-capita output when the income tax is completely eliminated. Assume that the government throws away all tax revenue to the ocean.

   Consider the following economy. A representative consumer solves

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L)^{1-\sigma}}{1-\sigma} L dt, \ \rho > 0, \sigma > 0, \ \sigma \neq 1,$$

   where ρ is a parameter, $C(t)$ is aggregate consumption at period t, and L is constant population. (There is no population growth.) The constraints are

$$Y(t) = K(t)^{\alpha}(A(t)L)^{1-\alpha}, \ 0 < \alpha < 1,$$
$$Y(t) = C(t) + I(t) + \tau Y(t), \ 0 \leq \tau < 1,$$
$$\dot{K}(t) = I(t) - \delta K(t), 0 < \delta < 1,$$
$$A(t) = A(0)e^{g_A t}, \ g_A \geq 0 \text{ and } A(0) \text{ given,}$$

where $Y(t)$ is aggregate output, $K(t)$ is aggregate physical capital stock, α is a parameter, $A(t)$ is the productivity exogenously growing at g_A, $I(t)$ is aggregate investment, τ is a flat tax rate on total income, and δ is a depreciation rate. Normalize $L = 1$. Eliminating $Y(t)$ and $I(t)$, the four constraints above are re-written as the following two constaints:

$$\dot{K}(t) = (1-\tau)K(t)^{\alpha}A(t)^{1-\alpha} - C(t) - \delta K(t),\ 0 < \delta < 1,$$
$$A(t) = A(0)e^{g_A t},\ g_A \geq 0 \text{ and } A(0) \text{ given.}$$

To eliminate one more constraint, define

$$k(t) \equiv \frac{K(t)}{A(t)},\ c(t) \equiv \frac{C(t)}{A(t)},$$

and

$$\eta \equiv \rho - (1-\sigma)g_A.$$

Then, we can re-write the problem as

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t}\frac{(c(t))^{1-\sigma}}{1-\sigma}Ldt$$

subject to

$$\dot{k}(t) = (1-\tau)\,(k(t))^{\alpha} - c(t) - (\delta + g_A)k(t).$$

Note: *For derivation of the above:*

$$\frac{\dot{K}(t)}{A(t)} = (1-\tau)\left(\frac{K(t)}{A(t)}\right)^{\alpha} - \frac{C(t)}{A(t)} - \delta\frac{K(t)}{A(t)}.$$
$$\dot{K}(t) = \dot{k}(t)A(t) + k(t)\dot{A}(t).$$
$$\dot{k}(t) = (1-\tau)\,(k(t))^{\alpha} - c(t) - (\delta + g_A)k(t).$$
$$\int_0^{\infty} e^{-\rho t}\frac{(C(t))^{1-\sigma}}{1-\sigma}dt = \int_0^{\infty} e^{-\rho t}\frac{(A(t)c(t))^{1-\sigma}}{1-\sigma}dt$$
$$=> \int_0^{\infty} e^{-\rho t}\frac{(c(t))^{1-\sigma}}{1-\sigma}e^{g_A(1-\sigma)t}dt$$
$$= \int_0^{\infty} e^{-(\rho-(1-\sigma)g_A)t}\frac{(c(t))^{1-\sigma}}{1-\sigma}dt$$

(a) Obtain the first-order conditions.
Hint:

- $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}h(x(t),u(t))dt$ s.t. $\dot{x}(t) = g(x(t),u(t))$, $x(0)$ given.
- $H(x(t),u(t),\lambda(t)) = h(x(t),u(t)) + \lambda(t)g(x(t),u(t))$.
- First-order conditions: $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.
(Don't forget that our question has η in place of ρ here!)

Answer: *From*

$$H = \frac{(c(t))^{1-\sigma}}{1-\sigma} + \lambda(t)\left[(1-\tau)\left(k(t)\right)^{\alpha} - c(t) - (\delta + g_A)k(t)\right],$$

the first-order conditions are

$$c(t)^{-\sigma} = \lambda(t),$$
$$\lambda(t)\left[(1-\tau)\alpha k(t)^{\alpha-1} - (\delta + g_A)\right] = \eta\lambda(t) - \dot{\lambda}(t).$$

(b) Derive the Euler equation. (That is, eliminate $\lambda(t)$ in the system of two equations you derived in (a). Then express it with original notations such as $K(t)$, $C(t)$, $A(t)$ and ρ, eliminating $k(t)$, $c(t)$ and η.)

Answer: *We know* $\dot{\lambda}(t)/\lambda(t) = -\sigma\dot{c}(t)/c(t)$. *So*

$$(1-\tau)\alpha k(t)^{\alpha-1} - (\delta + g_A) = \eta + \sigma\frac{\dot{c}(t)}{c(t)}.$$

Then,

$$(1-\tau)\alpha\left[\frac{K(t)}{A(t)}\right]^{\alpha-1} - (\delta + g_A) = \rho - (1-\sigma)g_A + \sigma\left(\frac{\dot{C}(t)}{C(t)} - \frac{\dot{A}(t)}{A(t)}\right).$$

So

$$\underbrace{(1-\tau)\alpha\left[\frac{K(t)}{A(t)}\right]^{\alpha-1}}_{r(t)} - \delta = \rho + \sigma\frac{\dot{C}(t)}{C(t)}.$$

(c) Consider the economy on the balanced growth path, in which all variables grow at constant rates. Show that on this balanced growth path, $Y(t)$, $C(t)$ and $A(t)$ all grow at the same rate (i.e., $g_Y = g_C = g_A$). Be as explicit as possible.

Answer: *(Sketch) The Euler equation in (b) implies that* $g_K = g_A$. *But the production function* $Y(t) = K(t)^{\alpha}(A(t)L)^{1-\alpha}$ *implies* $g_Y = \alpha g_K + (1-\alpha)g_A$. *With* $g_K = g_A$, *this implies* $g_K = g_A = g_Y$. *Then,* $\dot{K}(t) = I(t) - \delta K(t)$ *implies* $\frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)} - \delta$, *so* $g_I = g_K = g_A = g_Y$. *Finally,* $Y(t) = C(t) + I(t) + \tau Y(t)$ *implies* $g_C = g_I = g_K = g_A = g_Y$.

(d) Suppose $\tau = 0.400$, $\alpha = 0.320$, $\delta = 0.050$, $\rho = 0.040$, $\sigma = 2$, and $g_Y = 0.022$. What is the (constant) level of $K(t)/Y(t)$ on the balanced growth path?

Answer: *The Euler equation becomes*

$$\frac{(1-\tau)\alpha}{K/Y} - \delta = \rho + \sigma\frac{\dot{C}(t)}{C(t)}.$$

So

$$K/Y = \frac{(1-\tau)\alpha}{\delta + \rho + \sigma\frac{\dot{C}(t)}{C(t)}} = \frac{(1-0.4)0.32}{0.05 + 0.04 + 0.044} = 1.433.$$

(e) Suppose that τ had been 40% upto year $t = 2009$, and all of a sudden, the government eliminates the tax so that $\tau = 0$ thereafter. All parameter values exogenously given (α, δ, ρ, σ and g_A) are the same as before. Suppose that after several years of "transition", the economy has reached a new balanced growth path. What would be the new (constant) level of $K(t)/Y(t)$ on this new balanced growth path?

Answer: *We showed that on the balanced growth path, $g_C = g_Y = g_A$. So we can still use the expression for K/Y in (d) because this expression is in terms of exogenous parameters only:*

$$K/Y = \frac{(1-\tau)\alpha}{\delta + \rho + \sigma g_A} = \frac{0.32}{0.05 + 0.04 + 0.044} = 2.388.$$

(f) Now let us compare two scenarios. In Scenario 1, $\tau = 0.4$ is imposed forever. In Scenario 2, $\tau = 0.4$ had been believed to be imposed forever, but as in (e), all of a sudden, the government eliminates the tax in $t = 2009$ so that $\tau = 0$ thereafter. Suppose at least in $t = 2050$, Scenario 2's economy would reach a new balanced growth path (as described in (e)). Obtain the ratio of $Y(2050)$ in Scenario 2 to $Y(2050)$ in Scenario 1. That is, how much more output does this economy have in Scenario 2? (Hint: $Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\alpha/(1-\alpha)} A(t)$.)

Answer: *$A(t)$ is exogenously growing, so is the same between the two scenarios. Outputs are*

$$\text{Scenario 1: } (1.433)^{0.32/(1-0.32)} A(t) = 1.185A(t).$$
$$\text{Scenario 2: } (2.388)^{0.32/(1-0.32)} A(t) = 1.506A(t).$$

So $1.506/1.185 = 1.271$. *Scenario 2's output will be 27% higher.*

(g) Draw a figure that shows your result as clearly as possible. On the x-axis is time t. On the y-axis is output ($Y(t)$). Draw Scenario 1's output path. Draw Scenario 2's output path. (You don't have to draw the path on the "transition" before the new balanced growth path is reached.) Are growth rates different between these two scenarios? Then why might the government want to decrease the tax rate?

Answer: *In this model, the output turns out to grow exogenously at g_A on the balanced growth path, so the growth rate is the same regardless of τ. But the level of output is higher in Scenario 2. During the transition, Scenario 2 will enjoy higher (transitional) growth rates.*