Cross-Sectional Returns in a General Equilibrium: Is There a Momentum?*

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Abstract

The quantitative features of cross-sectional stock returns, such as the momentum and a predictive power of dividend yields on future returns, are not fully understood within a general-equilibrium asset-pricing framework. Especially, the momentum is often considered to be hard to reconcile with the efficient market hypothesis. I calibrate a model with two Lucas trees and the recursive utility. Two trees represent an individual stock (or portfolio) and the rest of the stock market, respectively. The recursive utility is used to provide more realistic predictions on the equity premium and a risk-free rate. The model's predictions on an average individual stock's dividend yield, CAPM and C-CAPM betas, and the gain from a diversification, match reasonably with the data. More importantly, the model is able to generate the momentum profits reported by Lewellen (2002), as well as a cross-sectional regression slope of returns on lagged dividend yields that is close to empirical observations.

Keywords: Asset Pricing, Lucas Tree, Lucas Orchard, Recursive Utility, Momentum, Dividend Yield

JEL Classifications: G12

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1. Introduction

The momentum effect in the stock market has been empirically documented by several studies. Jegadeesh and Titman (1993) report that buying last year's top 10% winners and short-selling the bottom 10% losers provides a 8.2% annual profit on average. As Fama and French (1996) report, the momentum does not seem to disappear even after the risk factors that determine the cross-sectional properties of returns are controlled for. At a first sight, this anomaly seems to be unable to reconcile with the efficient market hypothesis. That is, if the stock market is competitive, prices should adjust quickly so that the momentum strategies, which use already available information to form portfolios, do not provide significant profits seemingly unrelated to risk factors.[1]

To understand this puzzle better, consider a theoretical question. Are there any properties in a general-equilibrium asset-pricing framework that force the evolution of cross-

[1] This motivated a series of behavioral apporaches in which, for example, investors are assumed to underreact to firm-specific news, compared to market-level news. However, this assumption cannot explain the fact that the momentum arises more strongly for well-diversified portfolios than for individual stocks, and then vanishes at the market level. (See Lewellen (2002).)

sectional returns to exhibit a momentum? The representative investor wants to diversify her (wealth) portfolio. If a stock's realized dividend growth is higher than realized consumption growth, then the dividend now occupies a higher share of the consumption. This may act for or against the investor's diversification motives. That is, it may happen that the change makes the wealth portfolio less diversified, which will, other things being equal, discourage the investor to hold it. But a market equilibrium makes her happy to do so, by decreasing a market price of the stock so that there is no excess supply. That is, by the force of a market equilibrium, the stock now has a higher expected return, as a compensation for a less diversified wealth portfolio. On the other hand, if the new dividend share turns out to provide a more diversified wealth portfolio, then the market equilibrium will lower the stock's expected return. While this explanation is theoretically approached by Cochrane, Longstaff and Santa-Clara (2008), this paper shows that it can be *quantitatively* relevant.

The quantitative features of cross-sectional stock returns, regarding, for example, the momentum, a price-dividend ratio (or its inverse, a dividend yield) and its predictive power on future returns, CAPM and C-CAPM betas, and the gain form diversification, have been documented in many empirical studies. To understand them collectively in a theoretical framework, one needs to extend a model of a single Lucas (1978) tree to that of an orchard, incorporating many assets. But only recently researchers began to actively pursue an orchard model to connect its predictions to empirical characteristics. (See, for example, Menzly, Santos and Veronesi (2004) and Cochrane, Longstaff and Santa-Clara (2008).) The literature does not have enough quantitative predictions from the model, and even theoretical ones are limited to restricted environments on preferences or dividend processes.

The goal of this paper is to study both theoretical and quantitative properties of cross-sectional returns on individual stocks or portfolios, under the recursive utility (as in Epstein and Zin (1989) and Weil (1989)), with two Lucas trees, each representing an individual stock or a portfolio and the rest of the stock market. For preferences, I consider the recursive utility which separates a risk aversion and the intertemporal elasticity of substitution, providing relatively reasonable predictions on the risk premium and a risk-free rate. For dividend processes, I consider dividend growth which is independent and identically distributed (i.i.d.) over periods, which keeps the model more transparent. From this property of dividend growth, we know that all cross-sectional dynamics including the momentum come from the determination of equilibrium prices, and not from, for example, an autocorrelation of dividend growth.

The model is tractable, having only two states for each tree – "high" and "low" dividend growths. As in the calibration of Mehra and Prescott (1985), a model of only two states is simple, but captures most (if not all) stochastic dynamics. Solving the model numerically, I discuss whether quantitative predications match with the data. For example, the price-dividend ratio, CAPM and C-CAPM betas, and the gain from diversification match reasonably with the data.[2]

An important question is whether our "laboratory" based on this model can generate a momentum. I construct (fictional) portfolios that have stochastic properties of dividend growths that different from each other, so that they become close to the ones studied in empirical analyses. Those portfolios also provide reasonable expected value and volatility of consumption growth, as well as a regression slope of returns on lagged dividend yields reported by Menzly, Santos and Veronesi (2004). I show that the momentum profits, empirically reported by Lewellen (2002), can be generated from them. Further, as Lewellen (2002) finds as a source of the momentum, a cross-serial correlation between period-t return on one tree and period-$(t+1)$ return on another is calibrated to be negative. These results imply that the momentum does not necessarily contradicts the efficient market hypothesis or other assumptions in a general-equilibrium asset-pricing model, such as a rational investor and a complete market.

In our model, the diversification motives that generate the momentum are reflected in equilibrium prices. Therefore, if all risk factors are *completely* controlled for in an empirical analysis, the momentum should disappear. For example, Lewellen (2002) reports that a three-factor model of Fama and French (1993) largely explains the momentum in

[2] On the other hand, the model's predictions on the equity premium, a risk-free rate and the return volatility, can be further improved in a future research. But this comes from the original tree model with the recursive utility, not from adding additional trees to the orchard. The model can be improved by introducing more sophisticated environments, as in Bansal and Yaron (2004).

size-sorted and book-to-market-sorted portfolios.[3] While there is no consensus yet on this issue, we may be able to understand the momentum more clearly with more advanced techniques to control for risk factors.

1.1. Related Literature

The orchard model has not been explored enough, but its theoretical basis goes back to a general N-tree solution in Lucas (1978). Two papers that are most closely related to this paper are Cochrane, Longstaff and Santa-Clara (2008) and Martin (2009). Cochrane, Longstaff and Santa-Clara (2008) consider two trees under the log utility. A departure of this paper is to extend the preferences to more general recursive utility, in order to obtain more reliable quantitative predictions. Martin's (2009) extension considers constant-relative-risk-aversion (CRRA) preferences with rare disasters (as in Barro (2006)). The recursive utility in this paper can provide further insights on the CRRA because CRRA is a special case of the recursive utility.

Another paper that considers an orchard model is Menzly, Santos and Veronesi (2004). Their preferences are habit-formulated, while ours is recursive. Their dividend-growth volatility is time-dependent and designed to provide relatively convenient solutions, while ours is i.i.d. over time and provides transparent discussions. Their focus is mainly on the P/D ratio, while ours includes the momentum, betas and others. Other orchard models include Bossaerts and Green (1989), Bansal, Dittmar and Lundblat (2005), Santos and Veronesi (2006), and Pavlova and Rigobon (2007).

2. Preliminaries

2.1. Orchard

Consider a closed two-tree endowment economy without storage. Trees A and B provide D_t^A and D_t^B units of period-t consumption goods, respectively, as fruits or dividends. The

[3] Lewellen (2002) reports that industry-level portfolios still have the momentum even under the three-factor model.

sum of them becomes the consumption, i.e., $C_t = D_t^A + D_t^B$. Tree j's dividend growth for $j = A$, B is

$$\frac{D_{t+1}^j}{D_t^j} = \begin{cases} g_H^j \text{ with probability } \pi_H^j \\ g_L^j \text{ with probability } 1 - \pi_H^j \end{cases},$$

where $g_H^j > g_L^j$, independent and identically distributed over periods. The dividend growths are allowed to be correlated between the trees.

This two-tree orchard can be understood as a simplified model of an orchard with N independent and identically distributed trees, in which N is large. To be specific, suppose that there is an aggregate shock that affects the probability distribution of individual dividend growth. (Simply put, an individual tree is more likely to have higher dividend growth if there is a "good" aggregate shock, and vice versa.) Then, labelling one of N trees as tree A and the remaining $N - 1$ as tree B, we have the above two-tree orchard. The Appendix 1 provides a more detailed discussion.

Table 1 provides the states of this economy. A share of tree A's dividend at t is denoted by s_t:

$$s_t \equiv \frac{D_t^A}{D_t^A + D_t^B}.$$

Here, $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, $\pi_H^A = \pi_1 + \pi_2$ and $\pi_H^B = \pi_1 + \pi_3$. Consumption growth and s_t growth in Table 1 are easily obtained by applying $C_{t+1}/C_t = (D_{t+1}^A/D_t^A)s_t + (D_{t+1}^B/D_t^B)(1 - s_t)$ and $s_{t+1}/s_t = (C_{t+1}/C_t)(D_{t+1}^A/D_t^A)$. A key observation is that as s_t varies, the stochastic properties of $\frac{Y_{t+1}}{Y_t}$ and $\frac{s_{t+1}}{s_t}$ also vary. Hence, we know that each tree's price and expected return in an equilibrium will depend on s_t. In short, s_t serves as a state variable.

Table 1: States of the Economy

State	Prob.	A's dividend growth, D^A_{t+1}/D^A_t	B's dividend growth, D^B_{t+1}/D^B_t	Consumption growth, C_{t+1}/C_t	s_t growth, s_{t+1}/s_t
#1	π_1	g^A_H (High)	g^B_H (High)	$g^A_H s_t + g^B_H(1-s_t)$	$\frac{g^A_H}{g^A_H s_t+g^B_H(1-s_t)}$
#2	π_2	g^A_H (High)	g^B_L (Low)	$g^A_H s_t + g^B_L(1-s_t)$	$\frac{g^A_H}{g^A_H s_t+g^B_L(1-s_t)}$
#3	π_3	g^A_L (Low)	g^B_H (High)	$g^A_L s_t + g^B_H(1-s_t)$	$\frac{g^A_L}{g^A_L s_t+g^B_H(1-s_t)}$
#4	π_4	g^A_L (Low)	g^B_L (Low)	$g^A_L s_t + g^B_L(1-s_t)$	$\frac{g^A_L}{g^A_L s_t+g^B_L(1-s_t)}$

We now discuss the calibration of the orchard. According to the World Federation of Exchanges, there are 5,602 U.S. companies (excluding investment funds) in American SE, NASDAQ QMX and NYSE Euronext (U.S.), in 2008. We are interested in an average individual stock. Hence, our focus in the model is around $s_t = 0.02\%$. More broadly, the market capitalization of top 10 companies in NYSE Euronext (U.S.) is 20.1%, and NYSE Euronext (U.S.) occupies about 80% of the entire U.S. stock market in value. Hence, our broader focus is on $0 < s_t < 2\%$.

Since an individual stock is small, the quantitative features of "the rest of the stock market" (tree B) are close to the entire stock market. In Mehra and Prescott's (1985) calibration, net consumption growth has a mean of 1.8% and a standard deviation of 3.6%. The values $g^B_H = 1.06$ and $g^B_L = 0.98$, with $\pi^B_H = \pi^B_L = 1/2$, match with them. For tree A, I assume that the expected dividend growth is the same as for tree B (i.e., 1.8%). This is natural since tree A represents an *average* individual stock. We have to determine the values for two more parameters. One is the volatility of tree A's dividend growth. The other is a correlation between tree A's and tree B's dividend growths. To do this, I use the following two conditions. First, as discussed previously, if tree A and tree B can be understood as 1 and $N-1$ trees out of N independent and identically distributed trees, we have a restriction on parameters, as Appendix 1 shows.[4] Second, the expected return on tree A (to be computed in a later section) is the same as the expected return on tree B. This is, again, natural because tree A is an average individual stock. It turns

[4] This implies that we now only have to identify one parameter, ε, as Appendix 1 shows.

out that $g_H^A = 1.12$, $g_L^A = 0.92$, $\pi_1 = \pi_4 = 0.35$, and $\pi_2 = \pi_3 = 0.15$ satisfy these two conditions. We will confirm that the second condition is satisfied later.

The quantitative descriptions so far provide the following properties of the orchard. Figures 1-3 provide an illustration of these properties.

Properties of the Orchard: *(1) $E_t[C_{t+1}/C_t]$ is constant and $\sigma_t[C_{t+1}/C_t]$ is U-shaped on s_t.*

(2) $\{s_t\}$ has a mean reversion to 0. Also, the volatility of $\{s_t\}$ is inverse-U-shaped on s_t.

(3) As s_t increases from 0 to 1, the correlation between D_{t+1}^A/D_t^A and C_{t+1}/C_t increases from 0.4 to 1.

(4) Stationary distributions are $\pi_\infty(s) = 1\{s = 0\}$ and $\pi_\infty(s) = 1\{s = 1\}$.

In Property (1), $E_t[C_{t+1}/C_t]$ is constant because the expected dividend growth is identical between two trees. Also, $\sigma_t[C_{t+1}/C_t]$ is U-shaped because idiosyncratic risks can be diversified in a middle value of s_t. In Figure 1, $\sigma_t[C_{t+1}/C_t]$ is minimized when s_t is very low, close to 0. This is because tree A is more volatile than tree B, making itself less attractive. It is not surprising that the level of s_t of our interest, $s_t = 0.02\%$, provides a well-diversified portfolio. That is, our calibration focuses on an average individual stock, and the wealth portfolio is the best diversified by holding identical individual stocks. Property (2) is depicted in Figure 2. Here, $\{s_t\}$ tends to mean-revert to 0, even though it almost follows a random walk. It is more volatile around 0.5. Property (3) is depicted in Figure 3. Property (3) is straightforward from the fact that as s_t converges to 0, tree A's dividend converges to zero, and consumption converges to tree B's dividend. So the correlation between A's dividend growth and consumption growth is the same as the one between A's and B's dividend growths. On the other hand, as s_t converges to 1, tree A's dividend converges to the consumption, so the correlation converges to 1. Property (4) provides two stationary distributions, although $\pi_\infty(s) = 1\{s = 0\}$ is more interesting for our purposes.

Figure 1: s_t vs. $E_t\left[\frac{C_{t+1}}{C_t}-1\right]$ and $\sigma_t\left[\frac{C_{t+1}}{C_t}-1\right]$

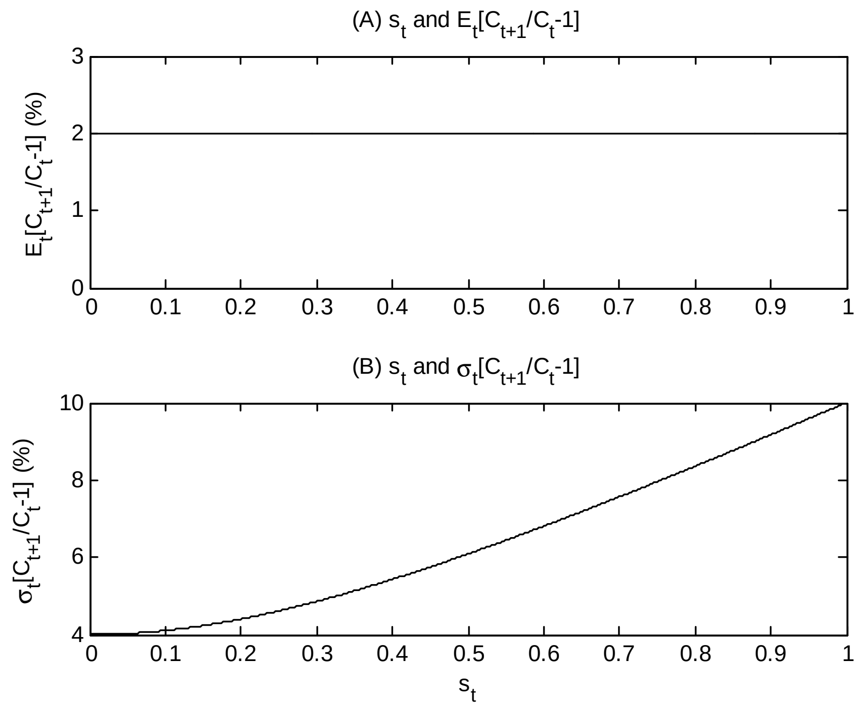


Figure 2: s_t vs. $E_t\left[s_{t+1}\right]$, $E_t\left[s_{t+1}\right]-s_t$ and $\sigma_t\left[s_{t+1}\right]$

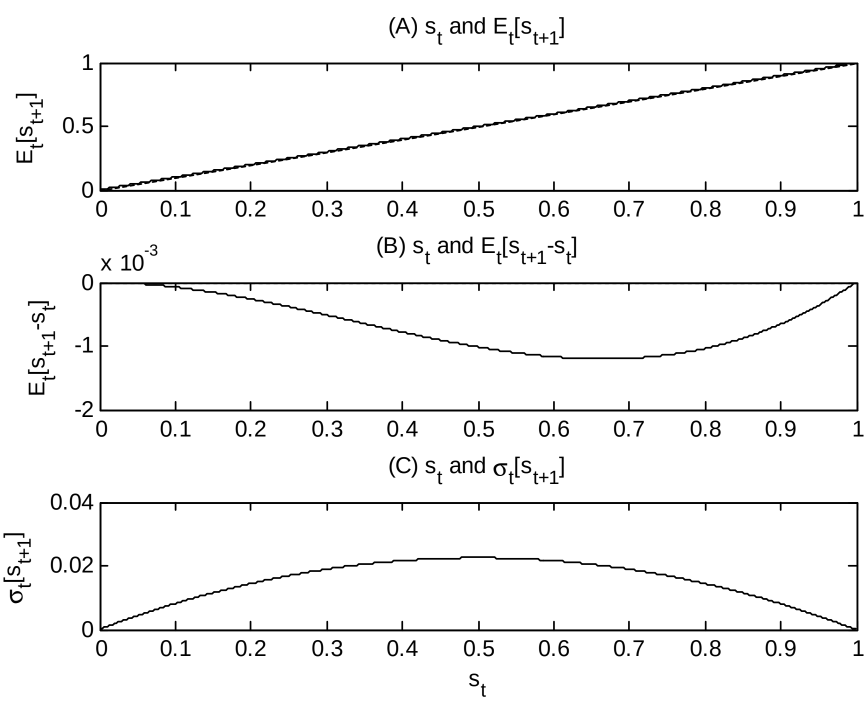
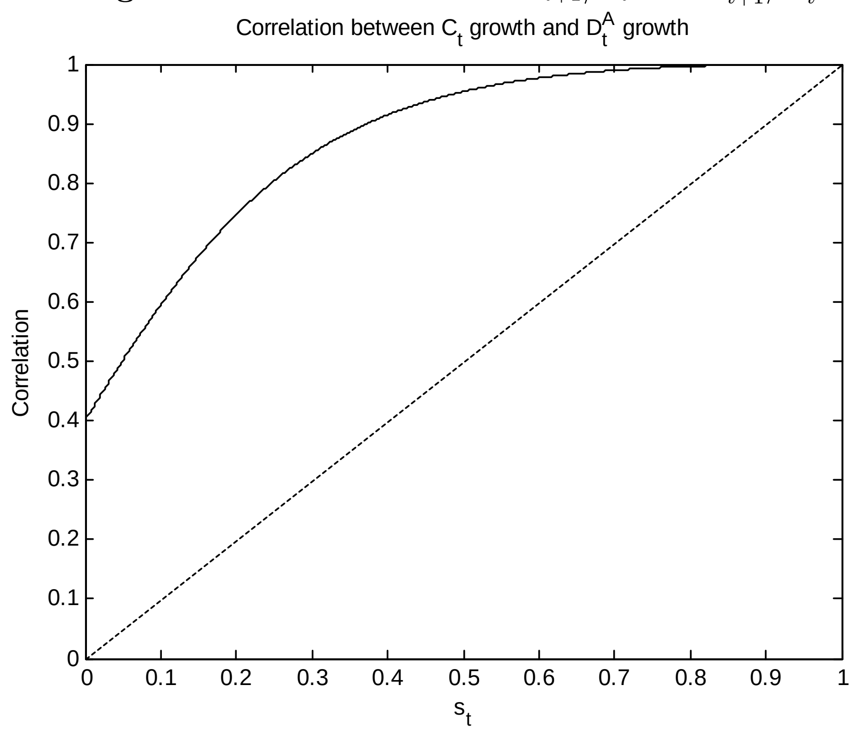


Figure 3: Correlation between C_{t+1}/C_t and D^A_{t+1}/D^A_t



2.2. Euler Equation

Since we have described the orchard, now we discuss how tree prices are determined. The representative consumer has the recursive utility described as

$$V_t = \left[(1-\beta)C_t^{1-\rho} + \beta(R_t(V_{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{1}$$

where

$$R_t(V_{t+1}) = [E_t V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}.$$

Here, V_t is the continuation utility index at period t, β is a time discount factor, ρ is an inverse of intertemporal elasticity of substitution, and γ is a risk aversion parameter. If $\rho = \gamma$, the preferences collapse to CRRA with parameter γ. The preferences provide the Euler equation for one-period asset j as

$$P_t^j = E_t\left[m_{t+1}X_{t+1}^j\right], \tag{2}$$

where

$$m_{t+1} = \beta \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}. \tag{3}$$

Here, P_t^j is asset j's price at period t and X_{t+1}^j is a random payoff at period $t+1$.

Two equations, (1) and (2), need to be rearranged for a calibration. Dividing both sides by C_t, we write (1) as

$$\frac{V_t}{C_t} = \left[(1-\beta) + \beta \left[E_t \left(\frac{C_{t+1}}{C_t} \frac{V_{t+1}}{C_{t+1}} \right)^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}}.$$

Define $V_t^* \equiv V_t/C_t$ to be the continuation utility index normalized by consumption. We look for a function, $V^*(s)$, so that V^* depends only on the state of the economy, s (in which a time subscript is omitted). Then, based on the states of the economy described in Table 1, we have to solve

$$V^*(s) = \left[(1-\beta) + \beta A^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}}$$

where

$$\begin{aligned} A = {} & \pi_1 \left[(g_H^A s + g_H^B(1-s)) V^*(\xi_1) \right]^{1-\gamma} + \pi_2 \left[(g_H^A s + g_L^B(1-s)) V^*(\xi_2) \right]^{1-\gamma} \\ & + \pi_3 \left[(g_L^A s + g_H^B(1-s)) V^*(\xi_3) \right]^{1-\gamma} + \pi_4 \left[(g_L^A s + g_L^B(1-s)) V^*(\xi_4) \right]^{1-\gamma} \end{aligned}$$

and

$$\begin{aligned} \xi_1 &= g_H^A s/(g_H^A s + g_H^B(1-s)), \quad \xi_2 = g_H^A s/(g_H^A s + g_L^B(1-s)), \\ \xi_3 &= g_L^A s/(g_L^A s + g_H^B(1-s)), \quad \xi_4 = g_L^A s/(g_L^A s + g_L^B(1-s)). \end{aligned} \tag{4}$$

With all parameter values given, the function, $V^*(s)$, can be numerically solved by value-function iterations. However, the function can be analytically evaluated at two extreme s-values, 0 and 1. To see this, notice that if $s=0$, the state variable of the next period, ξ_k, for all $k = 1$, 2, 3 and 4, is also zero, providing the solution,

$$V^*(0) = \left[\frac{1-\beta}{1-\beta[(\pi_1+\pi_3)(g_H^B)^{1-\gamma} + (\pi_2+\pi_4)(g_L^B)^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}}} \right]^{\frac{1}{1-\rho}}.$$

We can similarly show that

$$V^*(1) = \left[\frac{1-\beta}{1-\beta[(\pi_1+\pi_2)(g_H^A)^{1-\gamma}+(\pi_3+\pi_4)(g_L^A)^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}}}\right]^{\frac{1}{1-\rho}}.$$

These two analytical solutions can be used as initial guesses to accelerate the iteration.

We can also write (3) as a more convenient form with

$$m_{t+1} = \beta\left(\frac{V_{t+1}^*}{\left[E_t\left(\frac{C_{t+1}}{C_t}V_{t+1}^*\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}.$$

Since we know the stochastic properties of C_{t+1}/C_t (given in Table 1) and V_{t+1}^* (provided by the numerical solution discussed above), we can now use (2) to price any asset j.

As examples, we consider two assets, a risk-free asset and the wealth portfolio, under a special case in which $s_t = 0$. (Solutions for the prices under a general value of s_t are discussed in the next section.) If $s_t = 0$, the model collapses to a familiar one-tree model. First, consider a risk-free asset. A risk-free rate, r_t^f, depends on the state, s_t. We write it as a function, $r^f(s)$. If $s = 0$, then

$$\frac{1}{1+r^f(0)} = \beta\left[E_t\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right].$$

Second, consider the wealth portfolio. Divide both sides of (2) by C_t to have

$$\frac{P_t^W}{C_t} = E_t\left[\beta\left(\frac{V_{t+1}^*}{\left[E_t\left(\frac{C_{t+1}}{C_t}V_{t+1}^*\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(1+\frac{P_{t+1}^W}{C_{t+1}}\right)\right]. \tag{5}$$

Define $P_t^{W*} \equiv P_t^W/C_t$ to be the price-consumption ratio (P/C) of the wealth portfolio. A solution depends only on the state of the economy, s_t. Denote this solution by $P^{W*}(s)$. If $s = 0$, then

$$P^{W*}(0) = \frac{\beta\left[E_t\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{1-\rho}{1-\gamma}}}{1-\beta\left[E_t\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{1-\rho}{1-\gamma}}}.$$

From here, the return on wealth portfolio is obtained by

$$1 + r_{t+1}^{W} = \frac{C_{t+1} + P_{t+1}^{W}}{P_{t}^{W}} = \frac{\frac{C_{t+1}}{C_t} + \frac{C_{t+1}}{C_t} P_{t+1}^{W*}}{P_{t}^{W*}}.$$

So far, we have developed the pricing formulae for two assets under $s_t = 0$. In calibration, we assume $\beta = 0.97$. Following an estimation of Chen, Favilukis and Ludvigson (2009), we assume $\gamma = 57.5$ and $\rho = 0.6$. The assumed value of $\rho = 0.6$ implies an intertemporal elasticity of substitution of around 1.7. This value is reasonable, as discussed in Bansal and Yaron (2004, p. 1492). The assumed value of $\gamma = 57.5$ appears high for risk aversion. For example, Mehra and Prescott (1985) argue that a reasonable upper bound is around 10. However, in a relatively simple set-up like ours, high risk aversion matches better with the data. For example, Weil's (1989) estimate is above 40. Hansen and Jagannathan (1991) also document a high level.

An attractiveness of the recursive utility, compared to constant-relative-risk-aversion preferences, is more reasonable quantitative predictions on the risk-free rate and the equity premium, even without additional restrictions such as rare disasters. However, as we confirm here, it does not provide a perfect fit with the data. (See, for example, Donaldson and Mehra (2008) for a related discussion.) Inserting the assumed values, our equations provide analytical solutions of $r^f(0) = 1.5\%$ and $E_t[r_{t+1}^W | s_t = 0] = 5.5\%$, i.e., a risk-free rate of 1.5% and the expected return on wealth portfolio of 5.5%, when s_t is 0. Since our focus is on $s_t = 0.02\%$ which is close to $s_t = 0$, these values turn out to be good approximations for $r^f(0.02\%)$ and $E_t[r_{t+1}^W | s_t = 0.02\%]$. The returns data at Kenneth French's website and the CPI inflation data by Bureau of Labor Statistics suggest that in 1946-2008, the annual real interest rate on 1-month treasury bill is 0.6% on average. The annual value-weighted real rate of return on stocks is 7.7% on average.

2.3. Solving for Individual Returns

Now we consider the return on an individual tree for $0 < s_t < 1$. To do this, we first obtain the price-consumption ratio (P/C) and the price-dividend ratio (P/D). They vary according to the state variable, s_t. Why? If P/D is constant on the contrary, the investor with diversification motives will choose the most preferred level of s_t, no matter which level of s_t is given. The prices (i.e., P/D ratios) should be determined in a market equilibrium so that these two levels of s_t, desired and given, are always equated.

To quantitatively develop this result, we start from the Euler equation for tree A:

$$P_t^A = E_t \left[m_{t+1}(D_{t+1}^A + P_{t+1}^A) \right].$$

Divide both sides by C_t and define $P_t^{A*} \equiv P_t^A / C_t$ to be the P/C ratio for tree A. Then,

$$P_t^{A*} = E_t \left[\beta \left(\frac{V_{t+1}^*}{\left[E_t \left(\frac{C_{t+1}}{C_t} V_{t+1}^* \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(s_{t+1} + P_{t+1}^{A*} \right) \right],$$

which is analogous to (5). Again, we look for a function, $P^{A*}(s) = P_t^{A*}(s_t)$. Since we know V_t^* depends only on s_t, we can also write V_t^* as $V^*(s)$ for a given state, s. Further, we know C_{t+1}/C_t also depends on s_{t+1}, so we can write the consumption growth as $g^C(s)$ for a given state s. We eventually solve for a function, $P^{A*}(s)$, satisfying

$$P^{A*}(s) = \frac{\beta \sum_{k=1}^{4} \pi_k V^*(\xi_k)^{\rho-\gamma} g^C(\xi_k)^{1-\gamma} \left(\xi_k + P^{A*}(\xi_k) \right)}{\left[\sum_{k=1}^{4} \pi_k \left(g^C(\xi_k) V^*(\xi_k) \right)^{1-\gamma} \right]^{\frac{\rho-\gamma}{1-\gamma}}}, \tag{6}$$

where ξ_k for all k is defined in (4). If $s = 0$, then $P^{A*}(0) = 0$ is an obvious solution. An interpretation is that if $s = 0$, then tree A does not pay any dividends in the future, so its price is zero.

The function $P^{A*}(s)$ can be numerically solved by approximating (6) as a linear equation system. That is, write the equation as

$$P^{A*}(s) - \frac{\beta \sum_{k=1}^{4} \pi_k V^*(\xi_k)^{\rho-\gamma} g^C(\xi_k)^{1-\gamma} P^{A*}(\xi_k)}{\left[\sum_{k=1}^{4} \pi_k \left(g^C(\xi_k) V^*(\xi_k) \right)^{1-\gamma} \right]^{\frac{\rho-\gamma}{1-\gamma}}} = \frac{\beta \sum_{k=1}^{4} \pi_k V^*(\xi_k)^{\rho-\gamma} g^C(\xi_k)^{1-\gamma} \xi_k}{\left[\sum_{k=1}^{4} \pi_k \left(g^C(\xi_k) V^*(\xi_k) \right)^{1-\gamma} \right]^{\frac{\rho-\gamma}{1-\gamma}}}.$$

Now define a state vector, $\tilde{s} = (0.01, 0.02, ...0.99)$. (This is only for an illustration. In an actual computation, $\tilde{s}$ should be more densely defined, including 0.02% as a component.) Then, our equation system is expressed as $Ax = b$, where $x = (P^{A*}(0.01), P^{A*}(0.02), ..., P^{A*}(0.99))'$ and b consists of the right-hand side of the above equation evaluated at each element of

$\tilde{s}$. For A, define I to be an identity matrix with the same size as A. For each ith element of $\tilde{s}$, we compute the corresponding four values for ξ_1, ξ_2, ξ_3 and ξ_4 as follows. Consider ξ_1, and suppose that the jth element of $\tilde{s}$ is the closest to ξ_1. Then, add $-\beta\pi_1 V^*(\xi_1)^{\rho-\gamma} g^C(\xi_1)^{1-\gamma} / \left[\sum_{k=1}^4 \pi_k \left(g^C(\xi_k)V^*(\xi_k)\right)^{1-\gamma}\right]^{\frac{\rho-\gamma}{1-\gamma}}$ to element (i,j) of I. Do similarly for all ξ_2, ξ_3 and ξ_4, and for all elements of $\tilde{s}$, to obtain A. Then, apply the Gauss-Siedel method to solve for x.

Once we solve for this P/C ratio, the P/D ratio can be easily obtained as

$$\frac{P_t^A}{D_t^A} = \frac{P_t^{A*}}{s_t}.$$

What is the P/D ratio in a special case of $s_t = 0$? Dividing (6) by s, sending $s \longrightarrow 0$, and solving for the P/D ratio, we have the P/D ratio going to infinity. Then, the (net) return on tree A at period $t+1$, denoted by r_{t+1}^A, satisfies

$$1 + r_{t+1}^A \equiv \frac{D_{t+1}^A + P_{t+1}^A}{P_t^A} = \frac{C_{t+1}}{C_t}\frac{s_{t+1} + P_{t+1}^{A*}}{P_t^{A*}}. \tag{7}$$

For each state of s_t, four possible values for r_{t+1}^A can be obtained from this equation since we know stochastic properties of C_{t+1}/C_t, s_{t+1} and P_{t+1}^{A*}.

Since our calibration approach is now fully described, we are ready to discuss the calibration results.

3. Quantitative Implications

In this section, we discuss the quantitative implications of the calibrated model. We first obtain the P/C and P/D ratios for an average individual stock (with $s_t = 0.02\%$) and compare them to the data. Using these ratios, we then obtain the expected returns. The features such as CAPM and C-CAPM betas, the gain from the diversification, and the predictability of P/D ratio, are then discussed. Another feature, the momentum, is discussed in Section 4. Table 2 summarizes the main results of this section.

Table 2: Calibration: Assumed and Computed Values

(A) Assumed Values

Assumed Value	Definition (Reference)
$\pi_1 = \pi_4 = 0.35$, $\pi_2 = \pi_3 = 0.15$	State probabilities
$g_H^A = 1.12$, $g_L^A = 0.92$	Dividend growth of an individual stock
$g_H^B = 1.06$, $g_L^B = 0.98$	Consumption growth (Mehra and Prescott (1985))
$\beta = 0.97$	Time discount factor
$\gamma = 57.5$, $\rho = 0.6$	Preference parameters (Chen, Favilukis and Ludvigson (2009))

(B) Computed Values

Computed Value	Definition (Data)
$r^f(s_t = 0.02\%) = 1.5\%$	Risk-free rate (0.6%)
$P^{A*}(s_t = 0.02\%)/0.02\% = 30.0$	P/D for an average individual stock
$P^{B*}(s_t = 0.02\%)/99.98\% = 29.2$	P/D for the stock market (30, Menzly, Santos and Veronesi (2004))
$E[r_{t+1}^A \| s_t = 0.02\%] = 5.4\%$	Expected return on an average individual stock
$E[r_{t+1}^B \| s_t = 0.02\%] = 5.5\%$	Expected return on the stock market (7.7%)
(slope)= 0.08	Regression slope of expected excess return on CAPM betas (0.08, Cochrane (2005, Chap. 20))
$\sigma[r_{t+1}^A \| s_t = 0.02\%] = 10.1\%$	Return volatility of an average individual stock (49%, Statman (1987))
$\sigma[r_{t+1}^B \| s_t = 0.02\%] = 4.1\%$	Return volatility of the stock market (18%, Statman (1987))
$\frac{\sigma[r_{t+1}^B \| s_t=0.02\%]}{\sigma[r_{t+1}^A \| s_t=0.02\%]} = 0.4$	Relative volatility (0.4, Statman (1987))
(slope)= 1 for the original set-up, and $2.4 - 3.4$ for a 5-portfolio case	Regression slope of returns on lagged dividend yields (2.9, Menzly, Santos and Veronesi (2004))

3.1. P/C and P/D Ratios

The numerical solutions for the P/C and P/D ratios of tree A are illustrated in Figure 4. Our analytical solutions for them when $s_t = 0$ (i.e., a P/C ratio of 0 and a P/D ratio of infinity) are also confirmed here. The P/C ratio is increasing as s_t rises. This is natural because the tree delivers a higher share of consumption as s_t rises, so its price also rises. When $s_t = 0.02\%$ (which is of our interest), the P/D ratio is 30.0. Since we interpret tree

A as an average individual stock, this result is consistent with the P/D ratio of around 30, for industry-level portfolios or for the aggregate stock market. (See, for example, Menzly, Santos and Veronesi (2004), Figure 3, for years 1950-2002.) For an entire range of s_t, the P/D ratio is U-shaped, with its minimum at around $s_t = 0.5$ or 0.6. For lower values of s_t, the P/D ratio increases as s_t rises. One may interpret that a stock with lower s_t is a growth stock, having high valuations (and, as we will soon confirm, lower expected values).

Figure 5 depicts the P/C and P/D ratios of tree B. When $s_t = 0.02\%$, the P/D ratio is 29.2, which is, again, consistent with the data.

Figure 4: P/C and P/D Ratios of Tree A

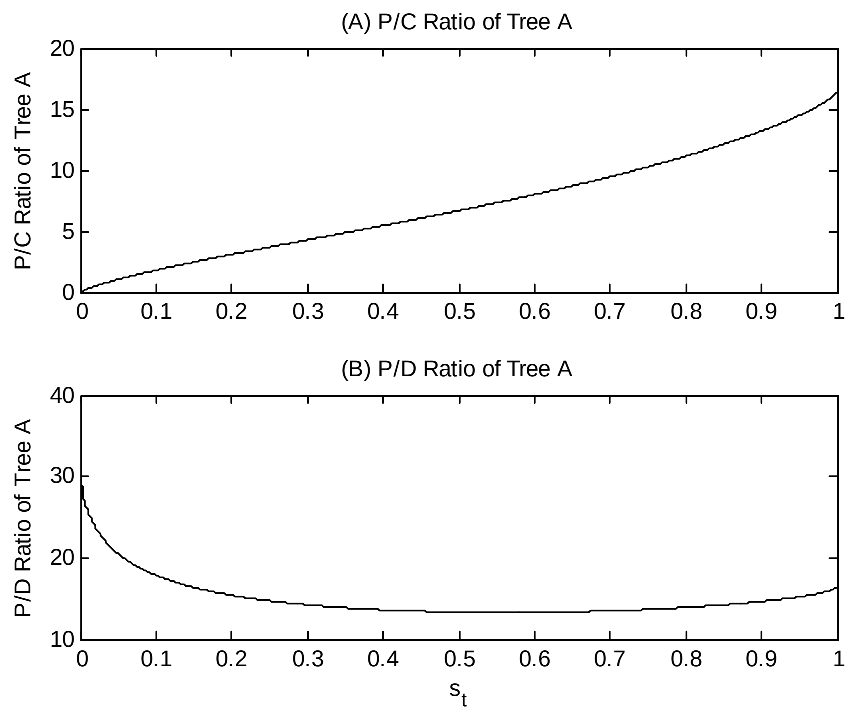
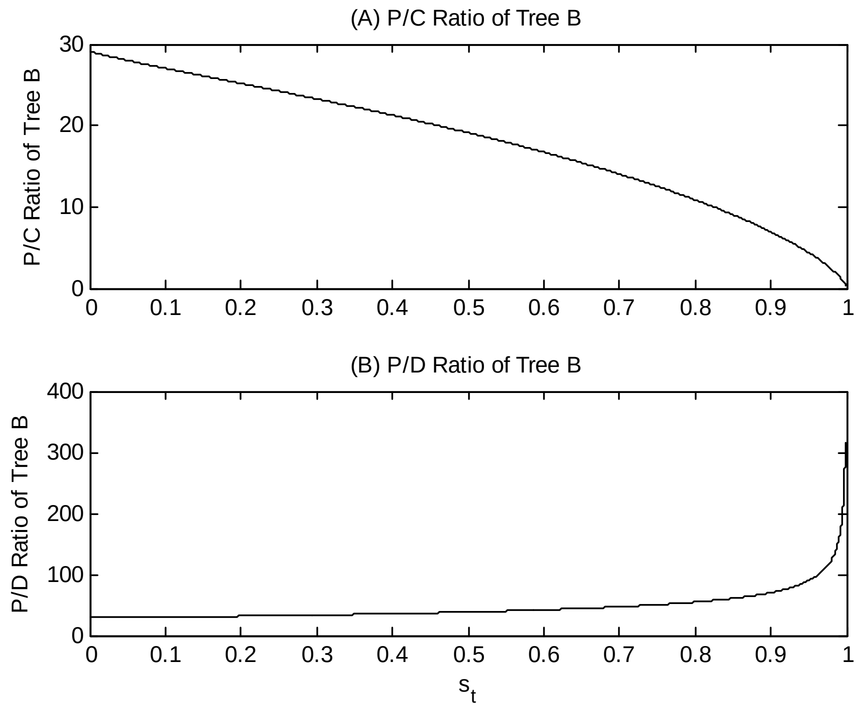


Figure 5: P/C and P/D Ratios of Tree B



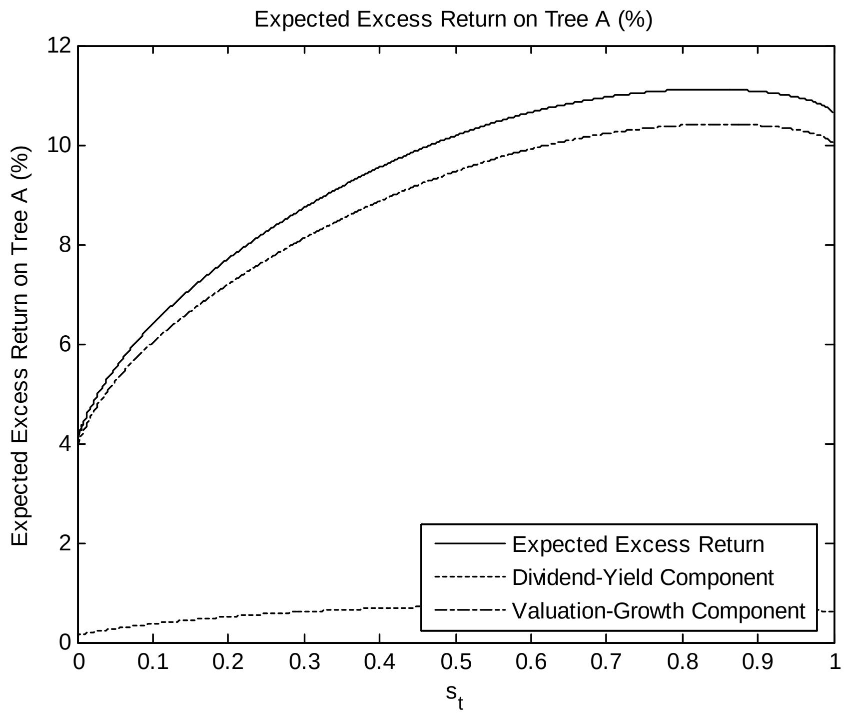
3.2. Expected Return on an Individual Stock

The expected return is the sum of the expected excess return and the risk-free rate. Here, we first focus on the expected excess return, and then extend our analysis to the expected return by adding the risk-free rate. To decompose the expected excess return, we start from the Euler equation, $1 = E_t[m_{t+1}(1 + r^A_{t+1})]$, and use $E_t[m_{t+1}] = 1/(1 + r^f_t)$ and (7) to have

$$E_t[r^A_{t+1}] - r^f_t = \underbrace{-(1 + r^f_t)cov_t\left[m_{t+1}, \frac{C_{t+1}s_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(A)}}\underbrace{-(1 + r^f_t)cov_t\left[m_{t+1}, \frac{C_{t+1}P^{A*}_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(B)}}.$$

The expected excess return is decomposed into two parts, (A) and (B). Since $C_{t+1}s_{t+1}$ is the next-period dividend from tree A and $C_t P^{A*}_t$ is tree A's price today, (A) is the compensation from the co-movement of a stochastic discount factor (m_{t+1}) and the dividend yield. Similarly, (B) is the compensation from the co-movement of a stochastic discount factor and the valuation growth. Both (A) and (B) can be numerically computed since we have the stochastic properties of m_{t+1}, C_{t+1}/C_t, s_{t+1} and P^{A*}_{t+1}. Figure 6 depicts this decomposition according to s_t.

Figure 6: Expected Excess Return on Tree A



The expected excess return is the lowest when $s_t = 0$, and then increases for the most range of s_t. This is because as s_t rises, tree A's dividend has an increasing share in consumption, strengthening the co-movement between tree A's dividend growth and consumption growth. This makes tree A less attractive, so it needs to be compensated by a higher expected excess return. When $s_t = 0.02\%$, the expected excess return is 3.9%, which is close to tree B's expected excess return, 4.0% (not reported in the figure). The decomposition shows that a larger part of an increase in expected excess return is a contribution from the valuation-growth component.

The risk-free rate is obtained from $E_t[m_{t+1}] = 1/(1+r^f_t)$. Adding this to the expected excess return gives the expected return on tree A, which is depicted in Figure 7. The risk-free rate is around 1.5% when $s_t = 0$. Then, it is decreasing as s_t rises. This is because the consumption volatility increases as s_t rises for most of the range, as in

Figure 1, strengthening the precautionary saving motives. The expected return on tree A is inverse-U-shaped. When $s_t = 0.02\%$, tree A's expected return is 5.4%, which is close to tree B's expected return, 5.5%, illustrated in Figure 8. When s_t is low, tree A's expected return is increasing in s_t. This implies that a positive shock on tree A's dividend increases (future) expected returns. This implies a positive autocorrelation, which is a possible source of the momentum. As s_t rises from 0 to 2%, the expected return also increases from 5.4% to 6.2%. Figure 8 is a counterpart figure for tree B. The expected return is the highest when s_t is 0. This is because when s_t is lower, the share of tree B's dividend in consumption is higher, which strengthens the co-movement between tree B's dividend growth and consumption growth. Again, this needs to be compensated by a higher expected excess return.

Figure 7: Expected Return on Tree A

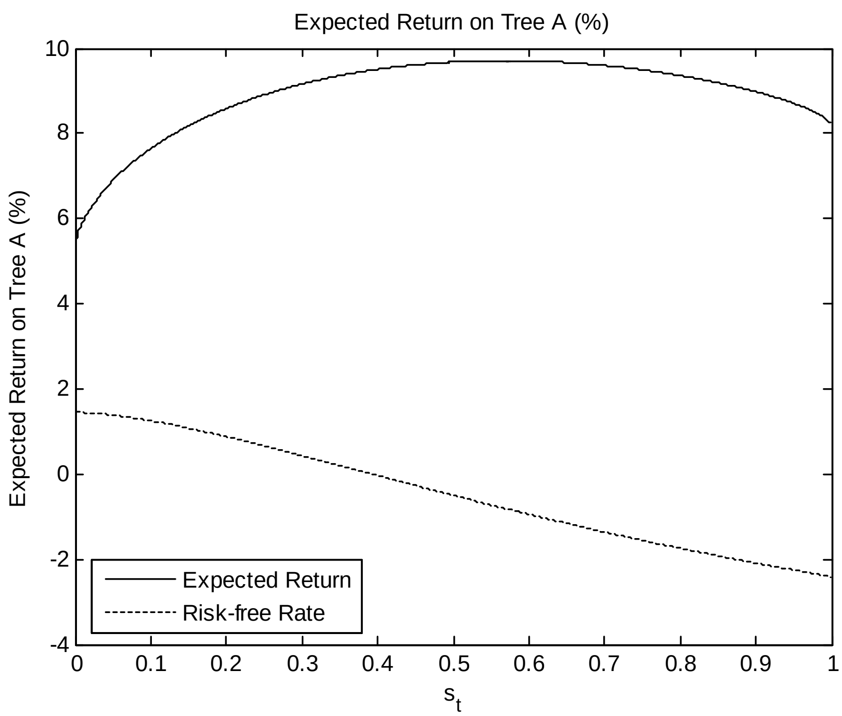
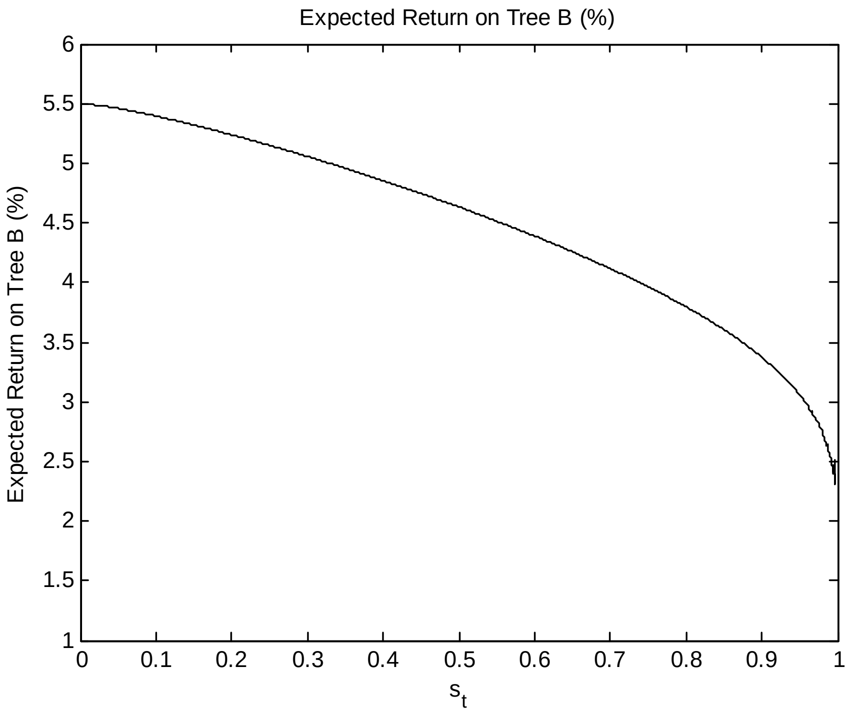


Figure 8: Expected Return on Tree B

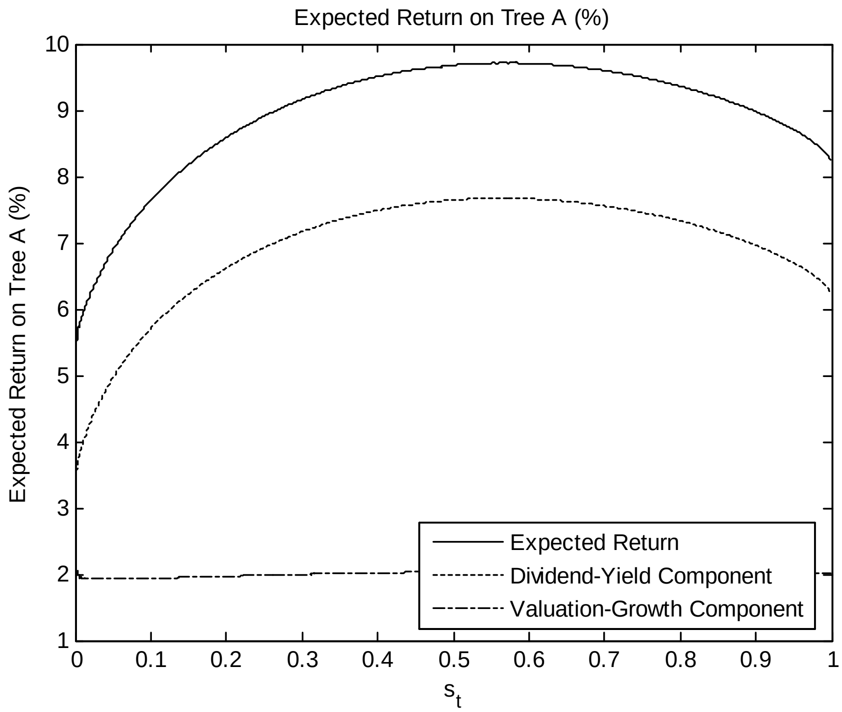


We can decompose the expected return in an alternative way. From the definition of r^A_{t+1}, we have

$$E_t\left[r^A_{t+1}\right] = \underbrace{E_t\left[\frac{C_{t+1}s_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(A)}} + \underbrace{E_t\left[\frac{C_{t+1}P^{A*}_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(B)}} - 1.$$

Here, (A) is the dividend-yield component and (B) is the valuation-growth component. They are depicted in Figure 9. The shape of the expected return is mostly driven by the dividend-yield component. To obtain the intuition, recall that the P/D ratio is U-shaped. The dividend-yield is the inverse of P/D ratio, so it is inverse-U-shaped. This determines the expected return, which is also inverse-U-shaped.

Figure 9: Expected Return on Tree A and Its Components



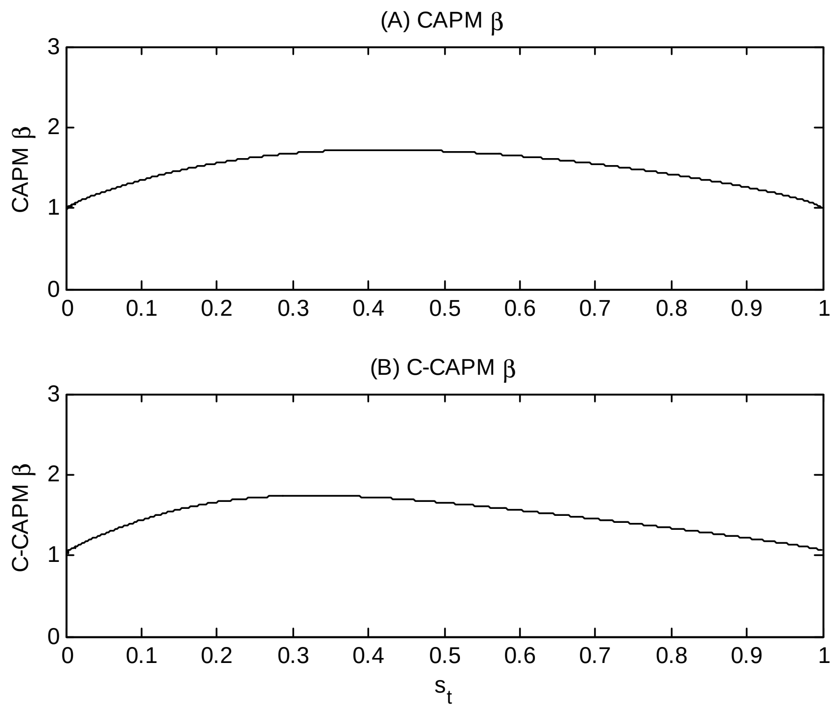
An issue of our model is that it does not provide an empirically observed size effect. That is, smaller stocks tend to have higher expected return in data, but the prediction of our model is the opposite. However, since we focus on an average individual stock, we have assumed that dividend growth does not depend on s_t, ignoring the fact that small firms tend to have more volatile dividend growths. To capture the size effect, we have to allow the dividend growth to depend on s_t. See, for example, Menzly, Santos and Veronesi (2004) for a similar approach. Or, we may consider several portfolios with different stochastic properties of dividend growth, and solve our model several times to approximate the solutions. We pursue this approach in Section 4.

3.3. CAPM β and C-CAPM β

The CAPM and C-CAPM betas for tree A, $\beta_t^{CAPM,A} \equiv cov_t(r_{t+1}^W, r_{t+1}^A)/var_t(r_{t+1}^W)$ and $\beta_t^{C-CAPM,A} \equiv cov_t(C_{t+1}/C_t, r_{t+1}^A)/var_t(C_{t+1}/C_t)$, also depend on the current state, s_t. Figure 10 depicts these betas. When s_t converges to 0, both betas converge to 1. Why?

In (7), we derived $1 + r^A_{t+1} = (C_{t+1}/C_t)\left[(s_{t+1} + P^{A*}_{t+1})/P^{A*}_t\right]$. But when s_t is 0, s_{t+1} is also 0. Also, since s does not change between t and $t+1$, P^{A*}_{t+1} will be the same as P^{A*}_t. So we have $1 + r^A_{t+1} = C_{t+1}/C_t$, which implies $\beta^{C-CAPM,A}_t$ is 1. By the same logic, we also have $1 + r^B_{t+1} = C_{t+1}/C_t$. But when s_t converges to 0, r^W_{t+1} converges to r^B_{t+1}. Hence, $\beta^{CAPM,A}_t$ is also one.

Figure 10: CAPM β and C-CAPM β of Tree A



When s_t is relatively low ($0 < s_t < 0.3$ or 0.4), betas increase as s_t rises. Here, higher betas are associated with higher levels of s_t, and hence with higher expected returns. This is consistent with CAPM and C-CAPM theories: Portfolios that give higher expected returns have higher betas. As s_t rises further from 0.3 or 0.4, betas decrease so that when s_t reaches 1, betas reach 1. This is because tree A converges to the wealth portfolio and tree A's dividend converges to the consumption as s_t approaches to 1.

Are the predictions consistent empirical results? As s_t rises from 0 to 2%, the CAPM beta increases from 1 to 1.1 in Figure 10. At the same time, the expected return increases from 5.4% to 6.2%, as previously discussed. Hence, the predicted regression slope of expected returns on CAPM betas is about 0.08. This value does not change by using expected *excess* returns. Empirically, Cochrane (2005, Chap. 20) reports a similar number. In an empirical figure reported by Cochrane (2005, Chap. 20), an increase in the CAPM beta from 0 to 1 is associated with an increase in the average excess return from 0 to about 8%. Hence, the slope is 0.08.[5]

3.4. Volatility of Returns

The volatility of the return can be decomposed as

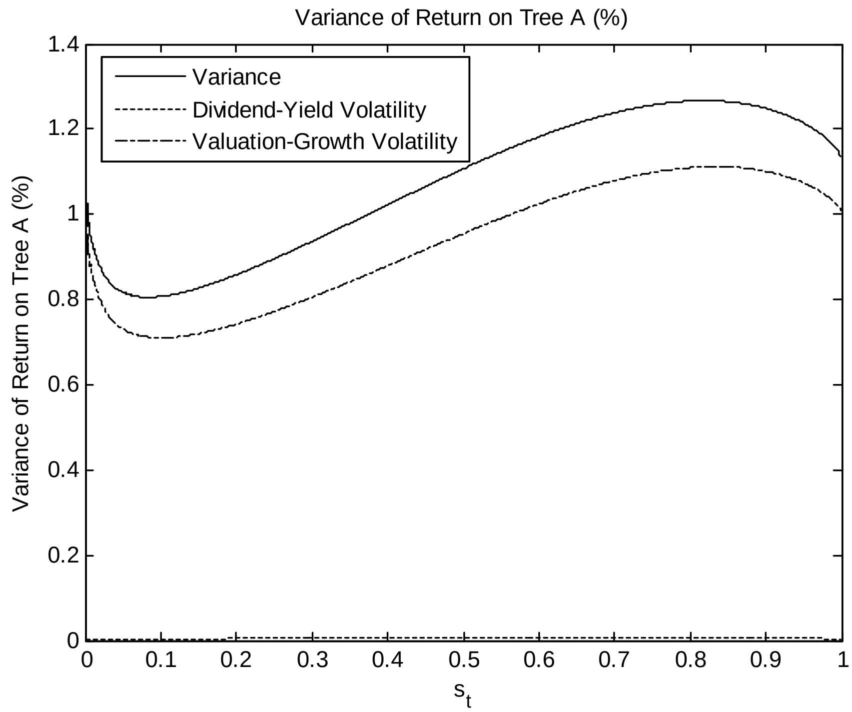
$$var_t[r^A_{t+1}] = \underbrace{var_t\left[\frac{C_{t+1}s_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(A)}} + \underbrace{var_t\left[\frac{C_{t+1}P^{A*}_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(B)}} + \underbrace{2cov_t\left[\frac{C_{t+1}s_{t+1}}{C_t P^{A*}_t}, \frac{C_{t+1}P^{A*}_{t+1}}{C_t P^{A*}_t}\right]}_{\text{(C)}},$$

where (A) is the variance of dividend yield, (B) is the variance of valuation growth, and (C) is the covariance term. This decomposition is depicted in Figure 11. Again, the valuation-growth component dominates. As $s_t \to 0$, the variance goes to infinity. Analytically, this is because the P/C ratio converges to zero while the variance depends on the inverse of P/C. As s_t grows from 0 to around 0.1, the variance decreases to its minimum at 0.8% (or a standard deviation of 9%). After that, the variance increases as s_t rises to around 0.8, and then it decreases again.

When $s_t = 0.02\%$, the standard deviations of tree A's and tree B's (not reported in Figure 11) returns are about 10.1% and 4.1%. Hence, the volatility of tree B's return relative to tree A's return is about 0.4. How do these predictions fit with the data? Statman (1987) reports that the volatilities of returns on individual stocks and on stocks in aggregate are 49% and 18%, respectively, making the relative volatility close to 0.4. Hence, the relative volatility is correctly predicted by the model. But there are excess (absolute) volatilities in data. The fact that our model fails to capture them, however, does not come from introducing multiple trees. This is a well-known character of an asset-pricing model with the recursive utility, in its simplest format.

[5] I skip a discussion on C-CAPM. There appears to be less concensus on C-CAPM estimations. For a recent development, see, for example, Lettau and Ludvigson (2001).

Figure 11: Variance of Return on Tree A



3.5. Predictive Power of the P/D Ratio

Here, we study the model's implications on a predictive power of P/D ratios (or dividend yields, which are inverses of P/D ratios) on future returns. Menzly, Santos and Veronesi (2004, p. 31, Table 5) run a pooled regression of industry-level returns on lagged dividend yields. They report a slope of 2.9.[6] Is our model able to generate this empirical relationship? At a first sight, the answer appears to be no. As s_t rises from 0 to 2%, the dividend yield increases from 3.3% to 4.2% in Figure 4. At the same time, the expected return increases from 5.4% to 6.2% in Figure 7. This implies a slope of around 1, which is lower than 2.9.

However, there are two important differences between our model and the data. First,

[6] Slopes vary according to the set-up. For time-series analyses, Fama and French (1988) run a regression of market-level returns on market-level dividend yields, reporting a slope between 2.6 and 7.7, depending on the time horizon. Menzly, Santos and Veronesi (2004) also report a similar time-series result for market-level returns.

our model focuses on an average individual stock (i.e., $s_t = 0.02\%$), while the data are based on well-diversified portfolios (i.e., say, $s_t = 0.1$ or 0.2). Second, our model treats all stocks identically, disregarding size, book-to-market and industry-level characteristics, while the portfolios in data have them. Then, how can we extend our model to reflect them? I now consider an economy with 5 portfolios with equal weights (e.g., $s_t = 0.2$). Portfolios are different in stochastic properties of dividend growth. Portfolio #1 has the lowest expected dividend growth but the dividend volatility is also the lowest (which corresponds to a portfolio of large stocks). Portfolio #5 has the highest expected dividend growth but the dividend volatility is also the highest (which corresponds to a portfolio of small stocks).

To be specific, each of the five portfolios has two possible states, "high" and "low" dividend growths. So there are $2^5 = 32$ states in the economy. There is an aggregate shock: An economy is "good" or "bad" with probability 1/2, i.i.d. over periods. If the economy is "good," each portfolio has "high" dividend growth with probability $1/2 + \varepsilon$ and "low" dividend growth with probability $1/2 - \varepsilon$. If the economy is "bad," each portfolio has "high" and "low" dividend growths with probabilities $1/2 - \varepsilon$ and $1/2 + \varepsilon$, respectively. Once the aggregate shock is realized, each portfolio's dividend growth is determined independently of each other and of periods. Portfolios #1, #2, #3, #4 and #5 have expected (gross) dividend growths, 1, 1.01, 1.02, 1.03 and 1.04. From here, I compare two set-ups. In the first, the portfolios have the distances between the expected growth and realized growth of 0.02, 0.04, 0.06, 0.08 and 0.10, respectively. For example, Portfolio #1's dividend growth is either 0.98 (low) or 1.02 (high). For Portfolio #5, it is either 0.94 (low) or 1.14 (high). Assuming $\varepsilon = 0.2$, the (net) consumption growth implied by this economy has an expected value of 2% and the volatility of 3.6%, similar to calibrated values in Mehra and Prescott (1985). In the second set-up, the distances are now 0.015, 0.030, 0.045, 0.060 and 0.075, respectively. I assume $\varepsilon = 0.4$ to have similar consumption volatility.

Our simple model of two trees and four economy-wide states is not able to analyze this economy with five portfolios. However, we can provide approximated solutions by solving the model five times, focusing on one portfolio at each. That is, at each time solving the model, two trees represent a portfolio and an (approximated) aggregate of the remaining four portfolios. Appendix 2 discusses the method in detail.

The results are summarized in Table 3. In set-up 1, the portfolios have the dividend yields ranging from 2.8% (portfolio #1) to 5.2% (portfolio #5). Expected returns range from 3.2% (portfolio #1) to 8.9% (portfolio #5). This suggests a regression slope of expected returns on dividend yields of around 2.4. In set-up 2, the slope becomes around 3.4. Therefore, our model is able to generate a predictive power of dividend yields on future returns, similar to the ones reported by Menzly, Santos and Veronesi (2004).

Table 3: Predictive Power of Dividend Yield

(A) Set-Up 1 ($\varepsilon = 0.2$)

Portfolio	g_H^j	g_L^j	Dividend Yield (inverse of P/D)	Expected Return
#1	1.02	0.98	2.8%	3.2%
#2	1.05	0.97	3.4%	4.6%
#3	1.08	0.96	3.9%	5.9%
#4	1.11	0.95	4.5%	7.4%
#5	1.14	0.94	5.2%	8.9%
			Slope:	2.4

(B) Set-Up 2 ($\varepsilon = 0.4$)

Portfolio	g_H^j	g_L^j	Dividend Yield (inverse of P/D)	Expected Return
#1	1.015	0.985	2.9%	3.3%
#2	1.040	0.980	3.2%	4.2%
#3	1.065	0.975	3.6%	5.7%
#4	1.090	0.970	4.0%	7.2%
#5	1.115	0.965	4.4%	8.4%
			Slope:	3.4

An issue is that our assumed values for g_H^j and g_L^j for all $j = 1, ..., 5$ are arbitrarily chosen, except that they provide a net 2% consumption growth on average. In other words, we showed that our laboratory is able to generate reasonable features of the predictability, but we do not know whether the laboratory has an environment that is close to the reality.

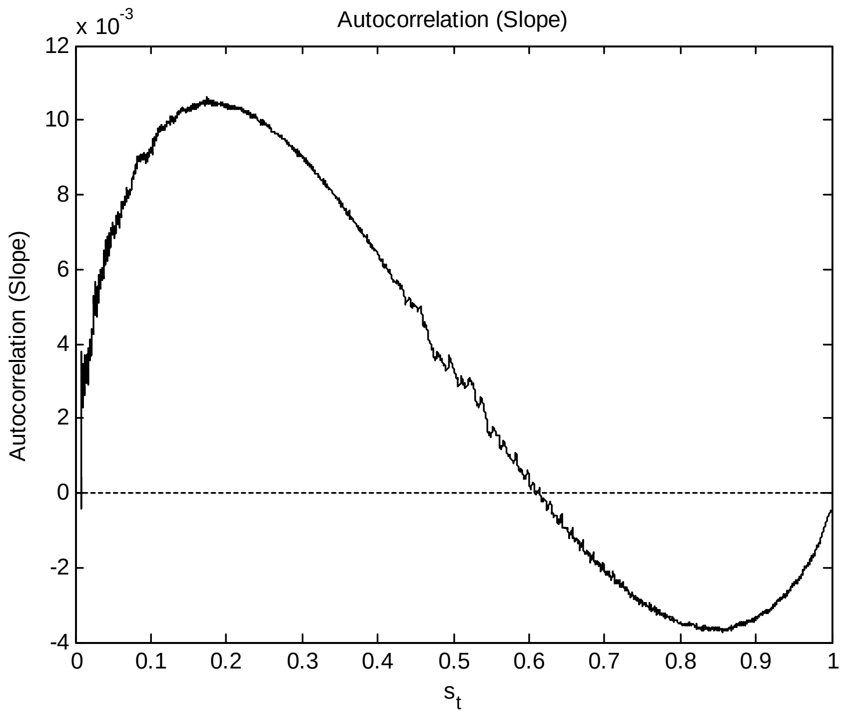
What are possible solutions to this issue? First, we can further investigate whether our laboratory can also provide other reasonable predictions. In the next section, we attempt to generate the momentum profits and compare them with the data. Second, we can fix the assumed values for g_H^j and g_L^j so that the expected values and volatilities of portfolio returns match reasonably with the observed ones for size-sorted, book-to-market-sorted, and industry-level portfolios. I do not pursue this approach because, as discussed earlier, our model does not match perfectly with the data, for expected returns and, especially, absolute volatilities of returns. That is, in Table 2, the return volatility on wealth portfolio is too low compared to the data, as in any application of an asset-pricing model with the recursive utility in its simplest format. Also, the equity premium and the risk-free rate are relatively reasonable, but still do not match perfectly with the data, making it difficult to use observations on portfolio returns. More sophisticated models as in Bansal and Yaron (2004) will be useful to pursue this approach.

4. Momentum

Lewellen (2002, p. 543) decomposes the momentum profit into three sources: (i) a positive autocorrelation of stock (or portfolio) j's returns, (ii) a negative cross-serial correlation between stock j's and stock i's returns ($j \neq i$), and (iii) the levels of unconditional expected returns. This section is organized as follows. First, we go back to the benchmark case in which tree A represents an average individual stock, and consider two sources, (i) and (ii), to understand whether there is a momentum for tree A. I find that both sources can be relevant. Second, we now consider a 5-portfolio economy as in Table 3. There are two reasons why we do this. First, it is not easy to generate the momentum from the benchmark because we have not introduced cross-sectional differences in individual stocks. That is, we are now adding "source (iii)" to obtain a more reasonable result. Second, Lewellen (2002) reports that well-diversified portfolios provide stronger momentum than individual stocks, so it is reasonable to focus on well-diversified portfolios. As Lewellen (2002) computes the momentum profits from the data, we follow the same way in our "laboratory."

We start by computing an autocorrelation of tree A's returns to study "source (i)". We obtain the slope in a regression of r^A_{t+2} on r^A_{t+1}, that is, $cov_t(r^A_{t+1}, r^A_{t+2})/var_t(r^A_{t+1}) = cov_t(r^A_{t+1}, E_{t+1}[r^A_{t+2}])/var_t(r^A_{t+1})$. Again, the momentum coefficient depends on the current state, s_t. Figure 12 depicts this slope according to s_t. When s_t converges to 0 or 1, the autocorrelation converges to 0 because the model collapses to a one-tree model and does not provide any diversification motives. Between 0 and 1, we see some autocorrelation, positive or negative. When $s_t = 0.02\%$, it is positive, although close to zero. The fact that it is positive implies that "source (i)" can be relevant. The fact that the autocorrelation is close to zero is not surprising because it converges to 0 when s_t approaches to 0. Still, if it is close to zero, then "source (i)" is perhaps not important for the momentum. In fact, Lewellen (2002) reports that the annual autocorrelations of returns on individual stocks and portfolios are not statistically different from zero, or in many occasions, even negative.

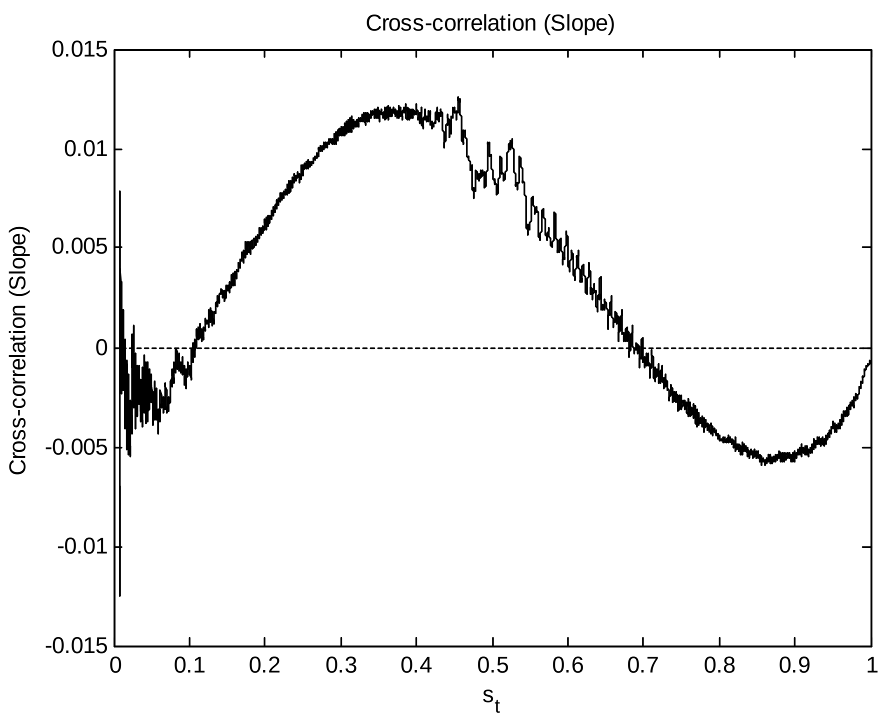
Figure 12: Autocorrelation of Tree A's return, $cov_t(r^A_{t+1}, r^A_{t+2})/var_t(r^A_{t+1})$



What does the model imply for "source (ii)"? Figure 13 shows the regression slopes of tree A's returns in $t + 2$ on tree B's return in $t + 1$, i.e., $cov_t(r^B_{t+1}, r^A_{t+2})/var_t(r^B_{t+1})$.

The nature of numerical solutions provides some noise in the figure, but we can see a clear trend that this cross-serial correlation starts at 0 when s_t is 0, remains negative when s_t is small, then increases to a positive value as s_t rises, then decreases back to a negative value, and finally recovers to 0 when s_t reaches 1. When s_t is small between 0 and 2%, the cross-serial correlation is negative. This implies that "source (ii)" is found in our model. Qualitatively, this is consistent with Lewellen's (2002) empirical result that cross-serial correlations of returns are mostly negative. Notice that in our model, the cross-serial correlation of *dividend growth* between trees A and B is zero, but the cross-serial correlation of returns is negative. That is, a negative cross-serial correlation of returns arises as a result of the diversification motives in an equilibrium.

Figure 13: Cross-Serial Correlation of Returns, $cov_t(r^B_{t+1}, r^A_{t+2})/var_t(r^B_{t+1})$



While we find that sources (i) and (ii) are present in our calibration, the model does not clearly distinguish the two sources. That is, it provides little guideline to determine whether Figure 13 is simply another way to present Figure 12, and vice versa. Eventually, the validity of our claim, that diversification motives can derive the momentum, can be more fundamentally understood by studying whether the model can generate the levels of momentum profits empirically observed. We now investigate whether our "laboratory" which generated reasonable regression slopes of expected returns on dividend yields in Subsection 3.5 is also able to generate the momentum profits. In Lewellen (2002), portfolio j's weight in year $t+1$ is formed by $w_{t+1}^j = (1/N)(r_t^j - r_t^W)$, where N is the total number of portfolios (between 5 and 25) and r_t^W implies a period-t return on wealth portfolio. Based on these weights, we consider a momentum strategy, investing \$1 long and \$1 short, respectively. From May 1963 to December 1999, this strategy generates an average profit between \$0.035 and \$0.056 (interpreted from monthly profits on value-weighted portfolios reported in Lewellen's (2002) Table 2).

We study the 5-portfolio economy previously considered. Hence, the model now includes "source (iii)" which we could not include in our benchmark calibration. A computation of the momentum profits is sketched in Table 4. Notice that there are 64 "scenarios": There is an aggregate shock, "good" and "bad", and once it is realized, there are $2^5 = 32$ states. The probability that the first scenario occurs, in which the economy is "good" and dividend growths for all five portfolios are "high", is $(1/2)(1/2+\varepsilon)^5 = 8.4\%$. For each scenario, we can approximate the stochastic movements of portfolio returns as in Subsection 3.5. For example, consider portfolio #1 as tree A and all others as tree B, in set-up 1 (i.e., $\varepsilon = 0.2$). Solving our two-tree model, four possible values of r_{t+1}^A are 5.6% (in which tree A has "high" dividend growth and the economy is "good"), 4.5% ("high" tree-A dividend growth and a "bad" economy), 1.5% ("low" tree-A dividend growth and a "good" economy), and 0.9% ("low" tree-A dividend growth and a "bad" economy). Also, our model also provides the values for $E[r_{t+2}^A | r_{t+1}^A, s_t = 0.2]$, which are 3.2%, 3.0%, 3.1% and 3.1%, respectively. These values can be obtained for all other portfolios. Since we know all possible returns in $t+1$ and all possible period-$(t+1)$ expected values of period-$(t+2)$ returns, we can now organize them for each scenario and construct Table 4. Then, we can finally obtain the expected momentum profits for all scenarios. In the first scenario, the profit becomes \$0.05. The average on all 64 scenarios is the one we compare with Lewellen's (2002) Table 2.

Table 4: (A Part of) Computation of Momentum Profits
(Set-Up 1: $\varepsilon = 0.2$)

Prob.	Agg. Shock	Dividend Growth	r^j_{t+1}	$E[r^j_{t+2} \mid r^j_{t+1}, s_t = 0.2]$	Momentum Portfolios	Expected Profits
8.4%	"Good"	P#1: "High" P#2: "High" P#3: "High" P#4: "High" P#5: "High"	$r^1_{t+1} = 5.6\%$ $r^2_{t+1} = 8.8\%$ $r^3_{t+1} = 11.8\%$ $r^4_{t+1} = 15.1\%$ $r^5_{t+1} = 18.1\%$	$= 3.2\%$ $(j = 1)$ $= 4.6\%$ $(j = 2)$ $= 5.9\%$ $(j = 3)$ $= 7.4\%$ $(j = 4)$ $= 8.8\%$ $(j = 5)$	Long: P#4: \$0.34 P#5: \$0.66 Short: P#1: \$0.67 P#2: \$0.32 P#3: \$0.01	\$0.05
...	...	...	...	...	...	...

The momentum profit predicted by our model is \$0.010 for set-up 1 ($\varepsilon = 0.2$) and \$0.059 for set-up 2 ($\varepsilon = 0.4$). Hence, the profit for set-up 2 is close to what Lewellen (2002) reports. One may argue that set-up 2's $\varepsilon = 0.4$ appears too high (even though I do not find a clear guidance on a reasonable value of ε). Then, it is true that in set-up 1 with a lower level of ε, the momentum profit is only 18%−29% of empirical levels. I argue that even set-up 1's result is not discouraging. Again, the return volatility in our model is low compared to the data, as shown in Table 2. But the momentum arises because returns are volatile, having different values across the states, as shown in Table 4. Our model predicts too low such volatility. Ultimately, a way to overcome this issue is to consider a model which perfectly predicts the risk premium, a risk-free rate and the volatility, which seems not easy (but probably not impossible) at the current point.

We can further improve our analysis on the momentum as follows. First, our solution is based on a two-tree economy, so we approximated the solutions for a five-portfolio economy by solving the model five times. It is possible to extend the model to consider a general N-tree economy for more straightforward discussions. Second, we conducted experiments for $\varepsilon = 0.2$ and $\varepsilon = 0.4$, but we will be able to generate more reasonable predictions with the parameter values that we may observe in data.

5. Concluding Remarks

I provided a numerical solution to a general-equilibrium asset-pricing model with a rational investor, the recursive utility, two trees, and a complete market. New predictions on cross-sectional properties of individual stocks or portfolios fit reasonably with the data. A sizable momentum profit is generated. A reasonable regression slope of returns on lagged dividend yields is also generated.

The study has several possible extensions. First, like other studies using the recursive model in its simplest format, our model does not provide perfect matches with the equity premium, a risk-free rate, and especially, the return volatility. Extending a tree-model to an orchard-model does not help to solve those puzzles. Related, our calibration is forced to assume a high level of risk aversion to provide plausible results. Introducing a more sophisticated set-ups, as in Bansal and Yaron (2004), may improve the performance.

Second, our assumption, i.i.d. dividend growth, is simple and transparent, but can be modified to match better with the data. For example, dividend growth can be allowed to depend directly on size, as in Menzly, Santos and Veronesi (2004). Also, Cohen, Polk and Vuolteenaho (2003) report that dividend growth is not i.i.d. over time, which can also be assumed in our model.

Third, our solution is not based on a steady state. We can introduce "birth" and "death" of firms to have a steady state in the solution, which provides a theoretical advantage. See a discussion in Cochrane, Longstaff and Santa-Clara (2008).

Fourth, it is possible to increase the number of trees and states to provide more straightforward discussions.

Fifth, other preferences, such as the habit formation, can be analyzed in a similar way. That is, we may extend Menzly, Santos and Veronesi (2004) to investigate whether the momentum is generated.

Sixth, the orchard model can be extended to consider the stock markets of multiple countries.

References and Notes

BANSAL, R., R. F. DITTMAR, AND C. LUNDBLAD (2005): "Consumption, Dividends, and the Cross Section of Equity Returns," *The Journal of Finance*, 60(4), 1639–1672.

BANSAL, R., AND A. YARON (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.

BARRO, R. J. (2006): "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121(3), 823–866.

BOSSAERTS, P., AND R. C. GREEN (1989): "A General Equilibrium Model of Changing Risk Premia: Theory and Tests," *Review of Financial Studies*, 2(4), 467–493.

CHEN, X., J. FAVILUKIS, AND S. C. LUDVIGSON (2009): "An Estimation of Economic Models with Recursive Preferences," Yale University Working Paper.

COCHRANE, J. H. (2005): *Asset Pricing: Revised Edition*. Princeton University Press, Princeton, NY.

COCHRANE, J. H., F. A. LONGSTAFF, AND P. SANTA-CLARA (2008): "Two Trees," *Review of Financial Studies*, 21(1), 347–385.

COHEN, R. B., C. POLK, AND T. VUOLTEENAHO (2003): "The Value Spread," *Journal of Finance*, 58(2), 609–642.

DONALDSON, J., AND R. MEHRA (2008): "Risk Based Explanations of the Equity Premium," in *Handbook of Equity Risk Premium*, ed. by R. Mehra, pp. 37–99. Elsevier Science, Amsterdam, The Netherlands.

EPSTEIN, L. G., AND S. E. ZIN (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

FAMA, E. F., AND K. R. FRENCH (1988): "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics*, 22(1), 3–25.

——— (1993): "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics*, 33(1), 3–56.

——— (1996): "Multifactor Explanations of Asset Pricing Anomalies," *Journal of Finance*, 51(1), 55–84.

HANSEN, L. P., AND R. JAGANNATHAN (1991): "Implications of Security Market Data for Models of Dynamic Economies," *Journal of Political Economy*, 99(2), 225–262.

Jegadeesh, N., and S. Titman (1993): "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, 48(1), 65–91.

——— (2001): "Profitability of Momentum Strategies: An Evaluation of Alternative Explanations," *Journal of Finance*, 56(2), 699–720.

Lettau, M., and S. C. Ludvigson (2001): "Resurrecting the (C)CAPM: A Cross-Sectional Test When Risk Premia Are Time-Varying," *Journal of Political Economy*, 109(6), 1238–1287.

Lewellen, J. (2002): "Momentum and Autocorrelation in Stock Returns," *Review of Financial Studies*, 15(2), 533–564.

Liu, L. X., and L. Zhang (2008): "Momentum Profits, Factor Pricing, and Macroeconomic Risk," *Review of Financial Studies*, 21(6), 2417–2448.

Lucas, Jr., R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46(6), 1429–1445.

Martin, I. (2009): "The Lucas Orchard," Stanford University Working Paper.

Mehra, R., and E. C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15(2), 145–161.

Menzly, L., T. Santos, and P. Veronesi (2004): "Understanding Predictability," *Journal of Political Economy*, 112(1), 1–47.

Pavlova, A., and R. Rigobon (2007): "Asset Prices and Exchange Rates," *Review of Financial Studies*, 20(4), 1139–1180.

Santos, T., and P. Veronesi (2006): "Labor Income and Predictable Stock Returns," *Review of Financial Studies*, 19(1), 1–44.

Statman, M. (1987): "How Many Stocks Make a Diversified Portfolio?," *Journal of Financial and Quantitative Analysis*, 22(3), 353–363.

Vuolteenaho, T. (2002): "What Drives Firm-Level Stock Returns?," *Journal of Finance*, 57(1), 233–264.

Weil, P. (1989): "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics*, 24(3), 401–421.

Appendix 1: From Many Individual Stocks to Two Trees

This appendix describes how we can use a two-tree orchard to analyze many identical stocks in an economy with an aggregate shock.

The orchard consists of N identically distributed trees where N is large. For simplicity, all trees have the same fruit shares at period t. In other words, D_t^j is identical for all $j = 1, 2, ..., N$. There is an aggregate shock, i.i.d over time. That is, the economy is "good" with probability 1/2, in which the dividend growth of tree j between t and $t+1$ follows

$$\frac{D_{t+1}^j}{D_t^j} = \begin{cases} g_H \text{ (high) with probability } 1/2 + \varepsilon \\ g_L \text{ (low) with probability } 1/2 - \varepsilon \end{cases},$$

for some constant ε. The economy is "bad" with probability 1/2, in which the dividend growth of tree j follows

$$\frac{D_{t+1}^j}{D_t^j} = \begin{cases} g_H \text{ (high) with probability } 1/2 - \varepsilon \\ g_L \text{ (low) with probability } 1/2 + \varepsilon \end{cases}.$$

The first out of these N trees is tree A. An aggregate of the remaining $N-1$ trees is tree B. How can we characterize tree B? If the economy is "good", then a fraction $1/2+\varepsilon$ of $N-1$ trees have high growths and a fraction $1/2-\varepsilon$ have low growths. Since N is large, this implies

$$\frac{D_{t+1}^B}{D_t^B} = \frac{\sum_{j=2}^N D_{t+1}^j}{(N-1)D_t^j} = (1/2+\varepsilon)g_H + (1/2-\varepsilon)g_L.$$

Similarly, if the economy is "bad",

$$\frac{D_{t+1}^B}{D_t^B} = (1/2-\varepsilon)g_H + (1/2+\varepsilon)g_L.$$

Then, there are 4 states: In state #1, tree A has high growth at g_H, and tree B also has high growth at $(1/2+\varepsilon)g_H + (1/2-\varepsilon)g_L$. And state #1 occurs with probability $(1/2)(1/2+\varepsilon)$. In state #2 which occurs with probability $(1/2)(1/2-\varepsilon)$, tree A has high growth at g_H, but tree B has low growth at $(1/2-\varepsilon)g_H + (1/2+\varepsilon)g_L$. In state #3 which occurs with probability $(1/2)(1/2-\varepsilon)$, tree A has low growth at g_L, but tree B has high growth at $(1/2+\varepsilon)g_H+(1/2-\varepsilon)g_L$. In state #4 which occurs with probability $(1/2)(1/2+\varepsilon)$, both tree A and tree B have low growths at g_L and $(1/2-\varepsilon)g_H + (1/2+\varepsilon)g_L$. This situation matches with Table 1 with $\pi_1 = \pi_4 = (1/2)(1/2+\varepsilon)$, $\pi_2 = \pi_3 = (1/2)(1/2-\varepsilon)$, $g_H^A = g_H$, $g_L^A = g_L$, $g_H^B = (1/2+\varepsilon)g_H + (1/2-\varepsilon)g_L$ and $g_L^B = (1/2-\varepsilon)g_H + (1/2+\varepsilon)g_L$.

The description so far justifies the two-tree orchard that we quantitatively discuss in the main text, with some restrictions we obtained so far (which are used for the calibration). In addition, it also implies that once we identify three constants, ε, g_H and g_L (as well as the number of trees, N), then the entire orchard is identified.

Appendix 2: From Five Portfolios to Two Trees

This appendix describes how we can use a two-tree orchard to approximate a 5-portfolio economy, where portfolios have different expected values and volatilities for dividend growth, in an economy with an aggregate shock.

The orchard consists of 5 trees. For simplicity, they have identical shares of 0.2 at period t. In other words, D_t^j is identical for all $j = 1, 2, ..., 5$. As in Appendix 1, there is an aggregate shock, i.i.d over time. That is, the economy is "good" with probability 1/2, in which the dividend growth of tree j between t and $t+1$ follows

$$\frac{D_{t+1}^j}{D_t^j} = \begin{cases} g_H^j \text{ (high) with probability } 1/2 + \varepsilon \\ g_L^j \text{ (low) with probability } 1/2 - \varepsilon \end{cases},$$

for some constant ε. The economy is "bad" with probability 1/2, in which the dividend growth of tree j follows

$$\frac{D_{t+1}^j}{D_t^j} = \begin{cases} g_H^j \text{ (high) with probability } 1/2 - \varepsilon \\ g_L^j \text{ (low) with probability } 1/2 + \varepsilon \end{cases}.$$

Notice a superscript j in g_H^j or g_L^j, implying that dividend growths are allowed to be different across portfolios.

Label tree #1 as tree A. Label an aggregate of the remaining 4 trees as tree B. Tree B's dividend growth has $2^4 = 16$ possible values, so our current model is not able to fully analyze the situation. To approximate it, we can compute the mean and standard deviation, and assume tree B's dividend growth has two possible values that satisfy them. Denote those values by g_H^B and g_L^B. Now the economy has 4 states: In state #1, tree A has high growth at g_H^1 and tree B also has high growth at g_H^B. It occurs with probability $(1/2)(1/2 + \varepsilon)$. The remaining 3 states can be similarly described. From here, we solve the model to compute the dividend yield and expected return for tree A, when $s_t = 0.2$.

We repeat the same procedure for tree #2, #3, #4 and #5.